ON AN ERGODIC THEOREM
FOR HOMOGENEOUS MARKOV CHAINS

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In this note we present a method for proving ergodic theorems that, in our view, is simpler than the approach contained in [1] (see also [2]).

Let \( \{X_n\}_{n=1}^\infty \) be a homogeneous Markov chain with transition function \( p(x, B) \), \( x \in X, B \in S \), where \((X, S)\) is a measurable space, and let \( p^{(n)}(x, B) \) be the transition function after \( n \) steps.

Assume that the chain \( \{X_n\} \) satisfies the following two conditions:

a) There exist an \( A \in S \), a nonnegative measure \( \varphi \) defined on \( AS \) and satisfying \( \varphi(A) > 0 \), and an \( n_0 \geq 1 \) such that for all \( x \in A \) and \( B \in AS \)
\[
p^{(n_0)}(x, B) \geq \varphi(B).
\]

b) For any \( x \)
\[
P_x\left( \bigcup_{1}^{\infty} \{X_n \in A\} \right) = 1.
\]

Here and in what follows, \( P_x \) denotes the probability in the space of sample functions under the assumption that the motion begins at the point \( x \). The conditions a) and b) are also imposed in [1], and they occur in equivalent form in [3] and [4].

Let us define a substochastic transition function \( \varphi(x, B) \) by the equalities \( \varphi(x, B) = \varphi(AB), x \in A \) and \( \varphi(x, B) = 0, x \in X \setminus A \). We set \( \omega(\cdot, \cdot) = p^{(n_0)}(\cdot, \cdot) - \varphi(\cdot, \cdot) \). For any measure \( \mu \) on \( S \) let
\[
\mu P(\cdot) = \int_X p(x, \cdot) \mu(dx).
\]

For any bounded \( S \)-measurable function \( f \) define
\[
Pf(\cdot) = \int_X f(x) p(\cdot, dx).
\]

The operators \( \Phi \) and \( \Omega \) corresponding to the transition functions \( \varphi \) and \( \omega \) are defined analogously. Let \( q(\cdot) = \sum_0^\infty \varphi \Omega^k(\cdot) \). Here \( \Omega^0 = E \), the identity operator. Denote by \( 1_B(\cdot) \) the indicator function of \( B \), and let \( \omega^{(n)}(x, B) = \Omega^n 1_B(x) \).

**Proposition.** The additive set function \( q(\cdot) \) is an invariant measure for the operator \( P \), i.e., \( qP \equiv q \). Moreover,
\[
q(A) = 1.
\]

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PROOF. Without loss of generality, it can be assumed that \( n_0 = 1 \). It is not hard to see that
\[
\varphi(A) \omega(k)(x, A) = \Omega^k \Phi_1 x(x),
\]
\[
\Phi_1 x = (P - \Omega^1) 1_x = (E - \Omega) 1_x.
\]
Consequently, for all \( N \geq 0 \)
\[
\varphi(A) \sum_{k=0}^{N} \omega(k)(x, A) = \sum_{k=0}^{N} \Omega^k (E - \Omega) 1_x(x) = 1 - \Omega^{N+1} 1_x(x).
\]
Let \( \{Y_n\}_{n=0}^{\infty} \) be a Markov chain with transition function \( \omega(x, B) \), and let \( n_k \) (respectively, \( N_k \)) be the number of times the chain \( \{Y_n\}_{n=0}^{\infty} (\{X_n\}_{n=0}^{\infty}) \) falls in \( A \) in \( k \) steps. It is not hard to see that \( P_x(n_k > m) \leq \alpha_m \) for any \( x \) and \( m \), where \( \alpha = 1 - \varphi(A) \). On the other hand, \( P_x(n_k < m) < P_x(N_k < m) \). In view of condition b), this means that
\[
\lim_{k \to \infty} P(n_k \leq m) = 0.
\]
Since
\[
\omega(k)(x, X) = P_x(n_k < m) + P_x(n_k > m),
\]
we get that \( \lim_{n \to \infty} \omega(n)(x, X) = 0 \). Returning to (3), we conclude that for any \( x \)
\[
\sum_{k=0}^{\infty} \omega(k)(x, A) = \frac{1}{\varphi(A)}.
\]
Integration of both sides of (4) with respect to \( \varphi \) gives (1).

Clearly, \( \varphi(\cdot) \) is countably additive. To prove that it is \( \sigma \)-finite we observe that for all \( m \) and \( N \)
\[
\sum_{k=0}^{N+m} \omega(k)(x, A) \geq \sum_{k=0}^{N} \int_{X} \omega(m)(y, A) \omega(k)(x, dy).
\]
Hence,
\[
\int_{X} \omega(m)(x, A) q_n(dx) \leq 1,
\]
where
\[
q_n(\cdot) = \sum_{k=0}^{N} \omega(k)(x, \cdot) \varphi(dx).
\]
Let \( f_m(x) = P_x(\min\{k \in A, k > 0\} = m) \) and \( B^n_m = \{x: f_m(x) \geq 1/n\} \). For any \( x \) we have \( \omega(m)(x, A) \geq f_m(x) \). By (5), this implies that \( q_n(B^n_m) \leq n \), and, consequently, \( q(B^n_m) \leq n \). On the other hand, \( X = \bigcup_{n,m} B^n_m \). Thus, \( q \) is a \( \sigma \)-finite measure.

It remains to verify that \( q \) is invariant. Obviously,
\[
\sum_{k=0}^{\infty} \varphi \Omega^k P = \sum_{k=0}^{\infty} \varphi \Omega^k (\Omega + \Phi) = \sum_{k=0}^{\infty} \varphi \Omega^k + \sum_{k=0}^{\infty} \varphi \Omega^k \Phi.
\]
Further,
\[
\varphi \Omega^k \Phi(\cdot) = \int_A \varphi(dx) \int_Y \omega(k)(x, dy) \varphi(y, \cdot) = \varphi(\cdot) \int_A \omega(k)(x, A) \varphi(dx).
\]
From (7) and (4) it follows that \( \sum_{k=0}^{\infty} \varphi \Omega^k \Phi = \varphi \). Returning now to (6), we conclude that
\[
\sum_{k=0}^{\infty} \varphi \Omega^k P = \sum_{k=0}^{\infty} \varphi \Omega^k, \text{ i.e., } qP = q, \text{ which is what was required.}
**Theorem.** If \( q(X) < \infty \), then for any \( x \)
\[
\lim_{n \to \infty} \sup_{B \in S} |p^{(n)}(x, B) - q(B)/q(X)| = 0.
\]

If \( q(X) = \infty \) and \( q(B) < \infty \), \( B \in S \), then for any \( x \)
\[
\lim_{n \to \infty} p^{(n)}(x, B) = 0.
\]

**Proof.** Without loss of generality it can be assumed that \( n_0 = 1 \). We make use of the identity
\[
(a + b)^n = a^n + \sum_{k=1}^{n} a^{n-k} b (a + b)^{k-1}, \quad n \geq 1,
\]
where \( a \) and \( b \) are elements of any ring.

Setting \( a = \Omega \) and \( b = \Phi \), we get
\[
P^n = \Omega^n + \sum_{k=1}^{n} \Omega^{n-k} \Phi P^{k-1}.
\]

Since for any \( k \) and \( j \)
\[
\int_X \omega^{(k)}(x, dy) \int_X p^{(j)}(y, B) \varphi(y, dz) = \omega^{(k)}(x, A) \int_A p^{(j)}(z, B) \varphi(dz),
\]

it follows from (8) that
\[
p^{(n)}(x, B) = \omega^{(n)}(x, B) + \sum_{k=1}^{n} \omega^{(n-k)}(x, A) p_{k-1}(B),
\]
where
\[
p_k(B) = \int_A p^{(k)}(z, B) \varphi(dz), \quad k \geq 1, p_0(B) = \varphi(AB).
\]

Integrating both sides of (9) with respect to the measure \( \varphi \), we have
\[
p_n(B) = \omega_n(B) + \sum_{k=1}^{n} \omega_{n-k}(A) p_{k-1}(B),
\]
where
\[
\omega_k(B) = \int_B \omega^{(k)}(x, B) \varphi(dx).
\]

This implies that if \( \sum_{k=1}^{\infty} \omega_k(B) < \infty \), then
\[
\lim_{n \to \infty} p_n(B) = \frac{1}{\mu} \sum_{k=0}^{\infty} \omega_k(B),
\]
where \( \mu = \sum_{k=1}^{\infty} (k+1) \omega_k(A) \), and \( 1/\mu = 0 \) when \( \mu = \infty \).

By definition, \( \sum_{0}^{\infty} \omega_k(B) = q(B) \). If \( q(X) < \infty \), then the convergence in (10) is uniform with respect to \( B \in S \).
In view of (2),

\[ \mu = \frac{1}{\varphi(A)} \sum_{k=0}^{\infty} (k+1) \varphi \Omega^k (E - \Omega) 1_x = \frac{1}{\varphi(A)} \sum_{k=0}^{\infty} \varphi \Omega^k 1_x = q(X)/\varphi(A). \]

The assertion of the theorem follows from (9)–(11) and (4).

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