2. Large deviations of sums of independent random variables: asymptotic formulas

2.1. Introduction. Let $X_1, X_2, ..., X_n, ...$ be sequence of i.i.d. random variables $X_i$ with a zero mean and an unit variance. Denote $S_n = \sum_{i=1}^{n} X_i$, $F(x) = \mathbb{P}(X < x)$, $F_n(x) = \mathbb{P}(S_n < x)$.

Yu.V. Linnik in a series of papers [1] has studied the conditions under which the probability $\mathbb{P}(S_n > x\sqrt{n})$ is approximated as $n \to \infty$ by $\mathbb{P}(Y > x)$, where $Y$ is the random variable with the standard Gaussian distribution, in the zone $0 < x < \Lambda(n)$, $\Lambda(n) \to \infty$ as $n \to \infty$.

Linnik’s method of proving is based on the direct analysis of behaviour of a characteristic function on the real axis and technically is very complicated, and conditions on the distribution of summand are not simple as well.

In 1965 I published paper [2], where, in particular, was suggested the absolutely different approach to proving limit theorems of Linnik type. It is sufficiently to notice that Linnik’s paper [1] numbers almost 50 pages in sum, whereas the proof of corresponding results in [2] takes fewer than 6 pages.

The conditions on the distribution of summands in [2] are also simplified significantly as compared with [1].

Since then the approach offered in [2] has been used repeatedly by different authors. In both [1] and [2] restrictions on the distribution of $X_i$ are imposed in the form of inequalities. In order to get asymptotics of large-deviation probabilities $\mathbb{P}(S_n > x\sqrt{n})$ for $x > \Lambda(n)$ one has to require that the function $\mathbb{P}(X > x)$ be regularly varying as $x \to \infty$ in one or another sense.

The first result in this direction was presented in my communication on the conference in Uzhgorod in October of 1959. The text of the communication is published in "Theory Probab. Appl." (see [3].) The complete proof is contained in [4] The condition concerning $\mathbb{P}(X > x)$ is formulated in this paper in terms of the characteristic function $\mathbb{E}e^{itX}$. It follows from this condition that

$$\mathbb{P}(X > x) \sim \frac{c}{x^\alpha}, \quad \alpha > 2. \quad (1)$$

As to the results of [1] and [2] in themselves they are given in the form of local theorems for the distribution $F_n(x)$ in the region $0 < x < o(\sqrt{n})$. It follows from these theorems that under conditions (1) large deviations of $S_n$ are formed mainly at the expense of one summand. Paper [2] is of interest not only because the new phenomenon is opened up in the latter. It should be also noticed that in this paper the complex analysis is applied successfully in the case when the Cramer condition is violated. Papers [3, 4] are discussed in greater detail in section 2.3. In the next my paper [5] (see, i.e. [6]) the asymptotic formula

$$1 - F_n(x) \sim n(1 - F(x)) \quad (2)$$

is deduced for $x$, going to infinity sufficiently fast, under condition that $F(x)$ is approximated on infinity by the distribution with a completely monotone density.

Remark that the new approach is used in [5] for the first time, which is based on the representation of the characteristic function of an approximating distribution in the form of the Cauchy - type integral on the positive semiaxis. Asymptotic formulae of type (2)
were later obtained by A.V. Nagaev [8, 10] under more general conditions, but by the pure
analytical method.

Asymptotic expression for $P(S_n > x)$ under conditions (1) with $\alpha > 1$ and $EX = 0$ is
obtained in my paper [9]. In 1969 A.V. Nagaev [10] deduced the asymptotics for $P(S_n > x)$
on the whole axis under assumption that

$$p(x) \sim e^{-x^\alpha}, \ 0 < \alpha < 1,$$

where $p(x)$ is the density of the distribution of the random variable $X_1$. The method of
proving in [10] is probabilistic. After a few years, in 1973, my paper [11] was published in
which the theorem on large deviations on the whole axis is proved under significantly more
general conditions than in [10]. The method of proving in [11] is, in contrast with [10], pure
analytical. The initial elements of this method are contained in [5]. The main result of [11]
is formulated also in [12].

2.2. Yu.V. Linnik’s results on large deviations. It was Yu.V. Linnik [1] who started
first to study the asymptotics of large deviation probabilities in the case when the well–known
Cramer condition

$$\int_{-\infty}^{\infty} e^{tx}dF(x) < \infty, \ t < t_0, \quad (3)$$

is violated. Here $F(x)$ is a distribution function of $X_1$. Linnik’s results are presented also
in [13] (chap. 11–14). Linnik has considered the distributions satisfying the condition

$$\int_{-\infty}^{\infty} e^{h(|x|)}dF(x) < \infty, \quad (4)$$

where $h(x)$ belongs to one of the classes which are described below.

Class I: the functions $h(x)$, satisfying the conditions

$$(\ln x)^2 + \xi_0 \leq h(x) \leq x^{1/2}$$

($\xi_0 > 0$ is as small as we please.) Furthermore $h(x)$ admits the representation

$$h(x) = \exp\{H(\ln x)\},$$

where $H(x)$ is a monotone differentiable function such that

$$H'(z) \leq 1, \ H'(z) \to 0 \text{ as } z \to \infty, \ H'(z)\exp\{H(z)\} > c_1z^{1+\xi_1}, \ \xi_1 > 0.$$  

Class II: the non decreasing continuous functions $h(x)$ satisfying the conditions

$$\rho_0(x)\ln x \leq h(x) \leq (\ln x)^2,$$

where $\rho_0(x)$ goes to infinity as slowly as one wish. Furthermore $h(x)$ can be represented in
the form

$$h(x) = M(x)\ln x = N(\ln x)\ln x,$$

where $N'(z) \to 0$ as $z \to \infty$.  

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Class **III**: the functions \( h(x) \), satisfying the conditions

\[
3 \ln x \leq h(x) \leq M \ln x,
\]

where \( M \geq 3 \) is some constant.

Linnik has proved that under condition (4) for \( 0 < x < \Lambda(n) \rho(n) n \)

\[
\lim_{n \to \infty} \frac{1 - F_n(x)}{1 - \Phi(x/\sqrt{n})} = 1,
\]

(5)

where \( \rho(n) \) tends to 0 as slowly as we please, and \( F_n(x) = \mathbb{P}(S_n < x) \). The bound \( \Lambda(n) \) depend on the class which \( h(x) \) belongs to.

For class **I** \( \Lambda(n) \) is defined by the equation

\[
h(\Lambda(n)) = n\Lambda^2(n).
\]

For class **II**

\[
\Lambda(n) = \sqrt{nh(x)} = \sqrt{M(n)n \log n}.
\]

For class **III**

\[
\Lambda(n) = \sqrt{n \log n}.
\]

**2.3. The alternative to Linnik’s method.** In 1950–1960, the problem of getting the bound of type

\[
\left| F_n(x\sqrt{n}) - \Phi(x) \right| < cg(x)\beta_3/n^{1/2}
\]

(6)

with an optimal function \( g(x) \), where \( \beta_3 = \mathbb{E}|X_1|^3 \), provoked general interest. In view of (2) the universal function \( g(x) \) can not decrease faster than \( 1/|x|^3 \). One may say that (6) is rough, but in return the collecting version of relation (2).

While trying to extend the equality (2) to a wider class of distributions by means of a preliminary truncation with subsequent applying the saddle-point method, I had met bounds for large-deviation probabilities of unknown earlier type which hold for sufficiently large \( x > \psi(n) \) (\( n \) is the number of summands). I failed to bring the initial plan to completion, however above-mentioned approach has made possible to obtain the bound (6) with \( g(x) = 1/(1 + |x|^3) \).

This new approach allowed to simplify and sharpen the asymptotic results by Linnik which were described above, namely, more extensive and simpler described class of functions than classes **I** and **II** by Linnik has been introduced. This class consists of the functions \( g(x) \) with a decreasing continuous derivatives satisfying the conditions

\[
0 < g'(x) < \frac{\alpha g(x)}{x}, \quad \alpha < 1, \quad x > B(g),
\]

(7)

and

\[
g(x) > \rho(x) \ln x,
\]

(8)

where \( \rho(x) \) goes to infinity as slowly as one wish when \( x \to \infty \), and \( B(g) \) is a positive constant depending on \( g \).

The next result has been obtained for this class.
Theorem 3.1. If
\[
\int_{-\infty}^{\infty} e^{g(|x|)} dF(x) < \infty,
\]
then for \(0 \leq x \leq \Lambda(n)\)
\[
\frac{1 - F_n(x)}{1 - \Phi(x/\sqrt{n})} = \exp\left\{ -\frac{x^3}{n^2} \lambda_{\alpha/(1-\alpha)} \left( -\frac{x}{\sqrt{n}} \right) \right\} (1 + o(1)),
\]
where \(\Lambda(n)\) is the solution of the equation \(x^2 = n g(x)\), and \(\lambda_{\alpha/(1-\alpha)}\) is the segment of the Cramer series containing \([\alpha/(1-\alpha)]\) first terms.

Condition (7) means that the function \(g(x)\) has the representation
\[
g(x) = c \exp\left\{ \int_{B(g)}^{x} \frac{\alpha(y)}{y} dy \right\}, \quad 0 < \alpha(y) < 1,
\]
which differs from the well-known representation for a slowly varying function in that \(\alpha(y)\), generally speaking, does not tend to 0 as \(y \to \infty\).

It is easily seen that the class of functions \(g(x)\) satisfying condition (7) with \(\alpha < 1/2\) and (8) contains classes I and II introduced by Linnik [1].

Theorem 3.2. If
\[
\int_{-\infty}^{\infty} |x|^m dF(x) < \infty,
\]
then for \(0 \leq x \leq \sqrt{\left( \frac{m}{2} - 1 \right) n \ln n}\)
\[
\frac{1 - F_n(x)}{1 - \Phi(x/\sqrt{n})} \to 1.
\]
Subsequently R. Michel [14] proved that the assertion of theorem 3.2 is the true in broader region of values \(x\). Condition (11) corresponds to condition (4) with \(h(x) = m \ln x\). Clearly, this function belongs to class III by Linnik. Notice that the bounds in Theorems 3.1 and 3.2 are sharper than these by Linnik.

Theorems 3.1 and 3.2, as well as nonuniform bound
\[
|F_n(x\sqrt{n}) - \Phi(x)| < c\beta_3/(1 + |x|^3)n^{1/2},
\]
are contained in my paper [2]. The latter has demonstrated efficiency of the truncation method. Before long, A. Bikelis [15] extended bound (12) to nonidentically distributed independent random variables. L.V. Osipov’s works on large deviation (see, e.g. [16]) are performed under influence of [2] as well.

The ideas which provide the basis for paper [2] have been used also in papers by A.A. Borovkov [17] and [18]. We return to them below.

2.4. Limit theorems on the whole axis. The first general result on the asymptotic behavior of sums i.i.d. random variables had been obtained by H. Cramer [19] who proved that under condition (5)
\[
\frac{1 - F_n(x\sqrt{n})}{1 - \Phi(x)} \sim \exp\left\{ \frac{x^3}{\sqrt{n}} \lambda\left( \frac{x}{\sqrt{n}} \right) \right\}
\]
(13)
if \( x = o(\sqrt{n}) \). Here \( \lambda(x) \) is the power series which is referred as Cramer’s series. Proving asymptotic formula (13) Cramer used the method of conjugated distributions.

If Cramer’s condition holds, then the moment generating function \( \int_{-\infty}^{\infty} e^{tx}dF(x) \) is analytical in the strip \( |Rez| < \delta \). This makes possible applying the saddle – point method to analyze large deviations. This method was used, in particular, by V. Richter [20] in proving local limit theorem on large deviations. If Cramer’s condition does not hold, then the characteristic function can not be extended into the complex plane as an analytical in some strip, and, consequently, the saddle – point method is not applicable, as well as the method of conjugated distributions. However, it turned out that it is possible to apply the complex analysis in the case when Cramer’s condition is violated.

For the first time, this was done in my paper [4] (the results of this paper were announced in [6]). It is supposed in this paper that moment generating function can be extended from the imagine axis into a multivalent function having the singularity at zero. It enables us to change the contour of integration. Let us formulate one of results obtained in [4] so that the reader can get more exact notion about its content. We suppose, as before, that \( X, X_1, X_2, ..., X_n, ... \) is a sequence of i.i.d. random variables with the distribution function \( F(x) \) and \( \text{Var}X < \infty \). Without loss of generality we may assume \( \E X = 0 \) and \( \text{Var}X = 1 \). Recall that \( F_n(x) := P(S_n < x) \).

Let the class \( D_0, 0 < \alpha \leq 1 \) of functions of the complex variable is defined as follows:

<table>
<thead>
<tr>
<th>( f(z) \in D_0 ) if it can be represented in the form</th>
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<tr>
<td>( cz^\alpha(1 + \psi(z)) ) for ( \alpha &lt; 1 ),</td>
</tr>
<tr>
<td>( cz \ln z\alpha(1 + \psi(z)) ) for ( \alpha = 1 ),</td>
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\( \psi(z) \) is a branch of of a multivalent function with the branch point at 0, \( \psi(z) \) being analytical in some circle \( |z| \leq A \), from which the segment \([0, A]\) is removed and \( \lim_{z \to 0} \psi(z) = 0 \) (the main branches are chosen for \( z^n \) and \( \ln z \)).

Let \( s \) and \( p \) be integers. We say that \( F(x) \) belongs to the class \( D_0^{sp} \alpha\beta \) if

\[
\int_{0}^{\infty} e^{itx}dF(x) = \sum_{k=0}^{s} \beta_k^+(it)^k + \phi_\alpha^+(it)(it)^s,
\]

\[
\int_{0}^{\infty} e^{itx}dF(x) = \sum_{k=0}^{p} \beta_k^-(it)^k + \phi_\beta^-(it)(it)^p,
\]


where \( \phi_\alpha^+ \in D_\alpha \), \( \phi_\beta^- \in D_\beta \), and

\[
\frac{dz^s}{dz^s} \phi_\alpha^+(z) = o(|z|^{-s}),
\]

\[
\frac{dz^p}{dz^p} \phi_\beta^-(z) = o(|z|^{-p}).
\]

It is easily seen that \( F(x) \in D_0^{sp} \alpha\beta \), if, for instance,

\[
F(x) = \begin{cases} 
\frac{c}{|x|^{p+\beta}}, & x < -1, \\
c, & |x| \leq 1, \\
1 - \frac{1-c}{x^{p+\alpha}}, & x > 1,
\end{cases}
\]

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where $0 < c < 1$ is some constant. Note that if $F(x) \in D_{\alpha\beta}^{sp}$, then Cramer’s condition (3) a fortiori does not hold since the analyticity of the function $\int_{-\infty}^{\infty} e^{zx} dF(x)$ in some strip $|Rez| < \delta$ follows from (3).

**Theorem 4.1.** Let $F(x) \in D_{\alpha\beta}^{sp}$, $s \geq 2, p \geq 2$ and there exists $n_0$ such that the distribution of the sum $X_1 + X_2 + ... X_{n_0}$ absolutely continuous and its density is bounded. Then, for $\alpha < 1$, $x \to \infty$, $x = o(\sqrt{n})$,

$$p_n(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} (1 + o(1)) + \frac{\Gamma(s + \alpha + 1) Im[c^+(1 - e^{2\pi i\alpha})]}{2\pi n(s + \alpha - 2) / 2 x^{s + \alpha + 1}} (1 + o(1)), \quad (17)$$

where $p_n(x)$ is density of the random variable $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$.

It is easily to check that the second summand in (17) is equal to the density $\sqrt{n}p(x\sqrt{n})$ of the random variable $x/\sqrt{n}$ multiplied by $n$. Thus, formula (17) can be written in the form

$$p_n(x) \sim \phi(x) + n^{3/2} p(x\sqrt{n}), x \to \infty, x = o(\sqrt{n}),$$

where $\phi(x)$ is the density of the standard normal law. It follows from the last formula that, for $b < a = o(\sqrt{n})$, $b \to \infty$,

$$F_n(a) - F_n(b) \sim \Phi\left(\frac{a}{\sqrt{n}}\right) - \Phi\left(\frac{b}{\sqrt{n}}\right) + n(F(a) - F(b)).$$

It is easy to derive from this fact that, for $x \to \infty, x = o(\sqrt{n})$,

$$1 - F_n(x) \sim 1 - \Phi\left(\frac{x}{\sqrt{n}}\right) + n(1 - F(x)). \quad (18)$$

Since under conditions of theorem 4.1,

$$p(x) \sim \frac{c(\alpha, s)}{x^{s + \alpha + 1}},$$

$$1 - F(x) \sim \frac{c(\alpha, s)}{\alpha + s} x^{-\alpha - s},$$

where

$$c(\alpha, c) = \frac{\Gamma(s + \alpha + 1) Im[c^+(1 - e^{2\pi i\alpha})]}{2\pi}.$$ 

It is easily seen, by using this fact, that

$$1 - \Phi\left(\frac{x}{\sqrt{n}}\right) = o(n(1 - F(x)))$$

if $x/\sqrt{n \ln n} \to \infty$. Thus, if (18) holds, then, for $x/\sqrt{n \ln n} \to \infty$,

$$1 - F_n(x) \sim n(1 - F(x)). \quad (19)$$

The result of such a type was obtained in my paper [5] under more general assumptions than in [4], namely,
Theorem 4.2. Let there exist the function \( \psi(x) \) integrable on the interval \((B, \infty)\) such that for any \( b > a > B \)

\[
V_{a \leq x \leq b} \left[ F(x) - G(x) \right] \leq \int_a^b \frac{\psi(x)}{x^2} g(x) dx,
\]

where \( V_{a \leq x \leq b} \) denotes a variation on the segment \([a, b]\), \( G(x) \) is the distribution on \([0, \infty)\) with the density

\[
g(x) = \int_0^\infty e^{-xu} \omega(u) du,
\]

i.e. the function \( g(x) \) is completely monotone in the sense of S.N. Bernshtein.

Assume, in addition, that the spectral function \( \omega \) has the second derivative satisfying the Hölder condition, and \( \omega(0) = 0 \). Then the representation (19) holds for \( x \) such that

\[
u u_x^2 = o(1),
\]

(20)

where \( u_x \) is the root of the equation

\[
u x e^{x u_x} (1 - G(x)) = 1.
\]

For example, if \( \omega(u) \sim u^\alpha, \alpha > 2, \) as \( u \downarrow 0 \), then \( 1 - G(x) \sim 1/\alpha x^\alpha \). It is easily seen that in this case condition (20) is fulfilled for \( x > \rho(n) \sqrt{n \ln n} \), where \( \rho(n) \) goes to infinity as slowly as we please. The bound obtained in such a way is close to \( \Lambda(n) \) in Theorem 4.2.

Suppose now that \( 1 - G(x) \sim e^{-x^\alpha} \) as \( x \to \infty \). Then condition (20) is fulfilled for \( x > \rho(n) n^{1-\alpha} \), where \( \rho(n) \to \infty \) as slowly as one wish. On the other hand, in view of Theorem 4.1 asymptotic formula (9) is valid for \( x < n^{1/2 - 2/\alpha} \).

Thus, we have the intermediate area \( n^{1/2 - \alpha} < x < n^{1/2 - 2/\alpha} \) in which the asymptotics of large-deviation probabilities is described neither by theorem 4.1 nor by the result of the paper [5].

The asymptotics of \( 1 - F_n(x) \) in the intermediate area \( n^{1/2 - \alpha} < x < n^{1/2 - 2/\alpha} \) was first investigated in the paper by A.V. Nagaev [10] under the condition \( F'(x) \sim e^{-|x|^\alpha}, 0 < \alpha < 1 \) as \( |x| \to \infty \). In contrast with [4,5] the pure probabilistic approach has been used in [10].

Proceed now to the description of results obtained in my paper [11]. Suppose that for \( x \to \infty \)

\[
1 - F(x) = e^{\chi(x)} (1 + o(1)),
\]

where \( \chi(x) \) is nonincreasing function which is defined for \( x > 0 \) and satisfies the following conditions

(i) \( \lim_{x \to \infty} x \chi'(x) / \log x = -\infty \),

(ii) there exists \( 0 < \alpha < 1 \) such that \( \alpha \chi(x)/x \leq \chi'(x) \),

(iii) \( l \chi''(x) \leq -\chi'(x)/x \leq L \chi''(x) \).

(iv) \( 0 \leq -\chi'''(x) \leq L_1 \chi''(x)/x \),

where \( l, L \) and \( L_1 \) are positive constants.

We need the additional conditions (iii), (iv) relative to Theorem 3.1 in order to justify the possibility of approximating the distribution \( F(x) \) on the infinity by the distribution
with an completely monotone density. Of course, we could impose this condition instead of (iii), (iv) in the very beginning, which would considerably simplify the presentation.

Assume that $$E|X_1|^{N(\alpha)} < \infty,$$
where \(N(\alpha) = [(3 - 2\alpha)/(1 - \alpha)]\). Introduce notation

$$K(u) = \sum_{k=2}^{N(\alpha)} \chi_k u^k,$$

where \(\chi_k\) are semi–invariant (cumulants) of the random variable \(X_1\). Let \(\lambda_\alpha(z)\) be the segment of the Cramer series consisting of the first \(N(\alpha) - 3\) terms.

Consider the equation

$$K'\left(-\chi'((1-u)x)\right) = ux/n$$

in the interval \(0 < u < 1\). Define \(\beta = \beta(x, n)\) as the least root of this equation (if, of course, it has a solution.) Further, let \(\Lambda(n)\) be the positive root of the equation

$$\chi(x) + x^2/n = 0.$$  \hspace{1cm} (22)

Notice that \(\Lambda(n)\) is well–defined since equation (22) has only one strictly positive root.

Put

$$P_1(x) = n \left(1 - \chi''((1 - \beta)x)\right)^{-1/2} \left(1 - F((1 - \beta)x)\right) \exp \left\{-(\beta x)^2/2n + \lambda_\alpha(\beta x/n)(\beta x)^3/n^2\right\},$$
$$P_2(x) = (1 - \Phi(x/n^{1/2})) \exp \left\{\lambda_\alpha(x/n)x^2/n^2\right\}.$$  \hspace{1cm} (21)

**Theorem 4.3.** Let \(F(x)\) satisfies conditions (i)–(iv). Then

$$P(S_n \geq x) = P_1(x)(1 + o(1))$$ \hspace{1cm} (23)

if

$$\lim_{n \to \infty} xn^{-1/(2-\alpha)} = \infty.$$  

If

$$\limsup_{n \to \infty} xn^{-1/(2-\alpha)} < \infty,$$

and

$$\limsup_{n \to \infty} n\chi''((1 - \beta)x) < 1,$$

then

$$P(S_n \geq x) = \left(P_1(x) + P_2(x)\right)(1 + o(1)).$$ \hspace{1cm} (24)

If \(\liminf_{n \to \infty} n\chi''((1 - \beta)x) \geq 1\) and \(\Lambda(n) \leq x \ or \ x \leq \Lambda(n)\), then

$$P(S_n \geq x) = P_2(x)(1 + o(1)).$$  \hspace{1cm} (25)
We see that conditions (iii) and (iv) are added in Theorem 4.3 as compared with Theorem 3.1. These conditions means that the function \( \alpha(y) \) in representation (10) for \(-\chi(x)\) satisfies the inequalities for \(-\chi(x)\)

\[
\frac{(1 - L^{-1})\alpha(x) - \alpha^2(x)}{x} < \alpha'(x) < \frac{(1 - l^{-1})\alpha(x) - \alpha^2(x)}{x}.
\]

Equation (21) which serves to compute the parameter \( \beta \) has been stated in my survey paper [11]. In particular, it has been showed there that \( \chi(x) = -x^{-\alpha} \), then, as it easily seen, \( \beta \sim 2\alpha n/x^{2-\alpha} \) for \( x/n^{1/2} \to 0 \). Hence, it follows that as \( x/n^{1/2} \to \infty \)

\[
(1 - (1 - \beta)^{n}) \to 0,
\]

i.e.

\[
F(x) - F((1 - \beta)x) = o(1).
\]

In addition under condition \( x/n^{1/2} \to \infty \)

\[
1 - \chi'^{(1 - \beta)x} \to 1 \quad \text{and} \quad \exp\left\{ - (\beta x)^2/2n + \lambda_n(\beta x/n)/(\beta x)^3/n^2 \right\} \to 1.
\]

As a result we conclude that for \( x/n^{1/2} \to \infty \)

\[
1 - F(x) \sim n(1 - F(x)).
\]

Incidentally the same bound we obtained above, starting from condition (20). It testify that the result obtained in [5] is sharp. Notice that the numerical solving of equation (21) involves no difficulties.

The asymptotic formulas which are close to those obtained in Theorem 4.3 are contained in the paper by Borovkov [18] (Theorem 2.1.) The latter assumes that \( \ln (1 - F(t)) = -l(t) = -t^{\alpha}L(t) \), where \( L(t) \) is a slowly varying function satisfying the condition

\[
L'(t) = o\left( \frac{L(t)}{t} \right)
\]

as \( t \to \infty \). In addition it is supposed that

\[
l''(t) = \alpha(\alpha - 1) \frac{l(t)}{t^2} (1 + o(1)).
\]

One may restate these conditions as follows: the functions \( a(x) \) and \( \varepsilon(x) \) in Karamata representation for \( L(x) \)

\[
L(x) = a(x) \exp\left\{ \int_1^x \frac{\varepsilon(y)}{y} dy \right\}
\]

satisfy the conditions

\[
a'(x) = o(x^{-1}), \quad a''(x) - a(x)\varepsilon'(x)x^{-1} = o(x^{-2}).
\]

These conditions are fulfilled, in particular, if

\[
a'(x) = o(x^{-1}), \quad a''(x) = o(x^{-2}), \quad \varepsilon'(x) = o(x^{-1}).
\]
Applying (28), we can represent \( l(x) \) in the form

\[
l(x) = a(x) \exp\left\{ \int_1^x \frac{\alpha(y)}{y} \ dy \right\}, \tag{29}
\]

where

\[
\alpha(y) = \alpha + \varepsilon(y).
\]

The principal difference between representations (10) and (29) lies in the fact that in the latter \( \lim_{x \to \infty} \alpha(x) = \alpha \), whereas in (10) \( \alpha(x) \) can oscillate on infinity between 0 and 1.

Notice also that smoothness conditions in Borovkov’s paper [18] are imposed just on \( \ln (1 - F(x)) \), not on the approximating function \( \chi(x) \) as in my paper [11]. On the other hand, only two derivatives of \( \ln (1 - F(x)) \) are required in [18], while three derivatives of \( \chi(x) \) are demanded in [11].

Thus, Borovkov’s conditions (26) and (27), generally speaking, are more restrictive than the conditions of Theorem 3.

The proof of Theorem 2.1 in [18] is based on the same probabilistic arguments as the proof by A.V. Nagaev [10].

References


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