**Lecture 4.**

**Lower and Upper Bounds for Global Optimum**

Consider an integer linear program

*How we can get an upper bound for ?*

*How we can get an lower bound for ?*

**Lagrangian Relaxation**

We rewrite this program as

(complicating constraints)

(nice constraints)

If we drop the complicating constraints, then we obtain a relaxation that is easier to solve than the original problem.

We assume that

Now for any nonnegative vector we consider the problem

:

where

The problem is called the *Lagrangian relaxation* of with respect   
to .

**Theorem.**  for all

**Proof**. Let us consider a feasible solution of . Note that is feasible for as well. Moreover, for because and

∎

**Lagrangian Dual**

The least upper bound available from the infinite family of relaxations is where is an optimal solution to

Problem is called the *Lagrangian dual* of with respect to   
the constraints

**Theorem**. where is a rational polyhedron, is a minimal convex polyhedron which contains all discrete points from

**Example 1**

|  |  |  |
| --- | --- | --- |
|  |  |  |
| s.t. |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |

All feasible points:

If we drop the constraint then we get the relaxed problem   
with optimal value 29. The set is finite and contains 8 points:

Lagrangian relaxation with respect to constraint is

s.t.

**How to solve the dual problem?**

Let be the optimal solution for the Lagrangian relaxation

Then

is the subgradient of function at the point We put

where is a suitable scalar coefficient.

If then

**Example 2**

The assignment problem with budget constraint

The is a set of jobs to be assigned to a set of workers, .   
 is the value of assigning worker to job ;

is the cost of training worker to do job ;

is training budget.

We wish to maximize the total value of the assignment subject to the budget constraint.

Optimization model:

s.t.

|  |  |
| --- | --- |
|  | (1) |
|  | (2) |
|  | (3) |
|  |  |

**How to choose a Lagrangian relaxation?**

Let us consider 4 variants:

1. Lagrangian relaxation with respect to (3):

|  |  |
| --- | --- |
|  |  |
| s.t. (1), (2) |

It is well-known assignment problem, integrality gap is , the linear programming relaxation has an integer optimal solution.

Hence,

1. Lagrangian relaxation with respect to (1) and (2)

|  |  |
| --- | --- |
|  |  |
| s.t. (3). |

It is well-known knapsack problem, integrality gap is nonnegative (often positive). Hence,

.

We can get better upper bound than in previous case.

1. Lagrangian relaxation with respect to (2)

|  |  |
| --- | --- |
|  |  |
| s.t. (1), (3). |

It is well-known multiple-choice knapsack problem. Integrality gap is nonnegative (often positive). Moreover, each feasible solution for is feasible in the . Hence,

We can improve the previous upper bound.

1. Lagrangian relaxation with respect to (2) and (3)

|  |  |
| --- | --- |
|  |  |
| s.t. (1). |

It is trivial to solve. For each we maximize and the corresponding is set to 1. Hence, the gap is and

Should we relax (1), (2), (3) at the same time or (1),(3)?

**Lagrangian Relaxation for the SPLP**

s.t.

Two ways to relax the problem.

What is the best one?

**Relaxation 1**

s.t.

Can we solve this problem in polynomial time?

**Relaxation 2**

s.t.

Can we find in polynomial time?

**Hometask 1.** The Capacitated Facility Location Problem

s.t.

Can we solve the Lagrangian relaxation problem with respect to   
 in polynomial time?

**Hometask 2.**

A company has two plants and three warehouses. The first plant can supply at most units and the second and most units of the same product. The sales potential at the first warehouse is , at the second warehouse , and at the third . The sales revenues per unit at the three warehouses are at the first, at the second, and at the third. The cost of manufacturing one unit at the plant and shipping it to warehouse is given in table. The company wishes to determine how many units should be shipped from each plant to each warehouse so as to maximize profit.

Table.

|  |  |  |  |
| --- | --- | --- | --- |
| From plant | To warehouse | | |
| **1** | **2** | **3** |
| **1** | 8 | 10 | 12 |
| **2** | 7 | 9 | 11 |

Create a mathematical model and solve it by Exel (*Поиск решения*).