On completely regular codes in Johnson graphs $J(2w+1,w)$ with covering radius 1

Sergey V. Avgustinovich, Ivan Yu. Mogilnykh

Sobolev Institute of Mathematics
Novosibirsk State University e-mails: avgust@math.nsc.ru, ivmog84@gmail.com

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**Code in graph**

*Code* $C$ in a graph $G$ is a collection of vertices of $G$.

*Distance* $d(x,y)$ between two vertices $x, y$ is the number of edges in the shortest path, connecting $x$ and $y$.

*Covering radius* $\rho$ of code $C$ in graph $G$ is a maximum distance from a vertex of graph to the code $C$:

$$\rho = \max \{ d(x, C) : x \in V(G) \}.$$
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**Covering radius** $\rho$ of code $C$ in graph $G$ is a maximum distance from a vertex of graph to the code $C$:

$$\rho = \max\{d(x, C) : x \in V(G)\}.$$
A code $C$ is called completely regular, if for any fixed $i$, $0 \leq i \leq \rho(C)$ the numbers $d_i^+(x)$, $d_i^0(x)$, $d_i^-(x)$ does not depend on choice of $x$ from $C_i$.

Intersection array of completely regular code $C$:
$\{d_1^-, \ldots, d_\rho^-, d_0^+, \ldots, d_{\rho-1}^+\}$. 
Completely regular code

\[ C_i = \{ x \in V(G) : d(x, C) = i \}, \ 0 \leq i \leq \rho. \]

For \( x \) from \( C_i \) denote with \( d_i^+(x), d_i^0(x), d_i^-(x) \) the number of vertices from \( C_{i+1}, C_i \) and \( C_{i-1} \) that are adjacent with \( x \).

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\[ \{ d_1^-, \ldots, d_\rho^-, d_0^+, \ldots, d_{\rho-1}^+ \}. \]
Johnson and Kneser graphs

Johnson graph $J(n,w)$

$V = \{ x \subset \{1, \ldots, n\} : |x| = w \}.$
$E = \{ (x, y) : |x \cap y| = w - 1 \}.$

Kneser graph $K(n,w)$

$V = \{ x \subset \{1, \ldots, n\} : |x| = w \}.$
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Completely regular codes in \( J(2w + 1, w) \) with covering radius 1
Completely regular codes in $J(2w + 1, w)$ and $K(2w + 1, w)$

From work by Neumaier \(^1\) we get:

**Statement**

A code $C$ in $J(2w + 1, w)$ with $\rho = 1$ is completely regular iff $C$ is completely regular code with $\rho = 1$ in $K(2w + 1, w)$.

Completely regular code in $J(9, 4)$ with array \( \{d_1^- = 15, d_0^+ = 6\} \)

exists iff exists completely regular code in $K(9, 4)$ with array \( \{d_1^- = 5, d_0^+ = 2\} \).
Completely regular codes in Johnson and Kneser graphs with $\rho = 1$

**One sporadic construction**

Completely regular codes with $\rho = 1$ from $(w-1) - (n, w, 1)$-designs

Completely regular codes in $J(9,4)$ with $\rho = 1$

Alltop’s extension constructions

**CRC in $K(9,4)$ with array** $\{d_1^- = 5, d_0^+ = 2\}$

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CRC in $K(9, 4)$ with array $\{d_1^-=5, d_0^+=2\}$

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On completely regular codes in Johnson graphs $J(2w+1, w)$ with...
CRC in $K(9, 4)$ with array $\{d_1^{-} = 5, d_0^{+} = 2\}$

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CRC in $K(9,4)$ with array $\{d_1^- = 5, d_0^+ = 2\}$

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C_1 & C_3 & C_5 \\
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1 & 0 & 0 & 4 \\
2 & 1 & 0 & 3 \\
3 & 0 & 1 & 3' \\
4 & 1 & 1 & 2e \\
5 & 0 & 2 & 2e \\
6 & 0 & 2 & 2' \\
7 & 1 & 2 & 1 \\
8 & 1 & 3 & 0 \\
9 & 0 & 3 & 1 \\
10 & 0 & 1 & 3e \\
11 & 1 & 1 & 2' \\
\end{array}
\]

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\begin{array}{cccccccccccc}
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2 & * & * & * & * & * & * & 0 & 2 & 0 & 0 \\
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\end{array}
\]
CRC in $K(9, 4)$ with array $\{d_1^- = 5, d_0^+ = 2\}$

Completely regular code in Kneser graph $K(9, 4)$ with intersection array:

$\{d_1^- = 5, d_0^+ = 2\}$

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Completely regular codes in Johnson and Kneser graphs with $\rho = 1$

One sporadic construction

Completely regular codes with $\rho = 1$ from $(w-1)-(n, w, 1)$-designs

Completely regular codes in $J(9,4)$ with $\rho = 1$

Alltop’s extension constructions

**CRC in $K(9,4)$ with array $\{d_1^{-} = 5, d_0^{+} = 2\}$**

Completely regular code in Johnson graph $J(9,4)$ with intersection array:

$$\left\{ d_1^{-} = 15, \ d_0^{+} = 6 \right\}$$
Completely regular codes from \((w - 1) - (n, w, 1)\)-designs

**Theorem (Martin ’98)**

Any simple \((w - 1) - (n, w, \lambda)\)-design is completely regular in \(J(n, w)\) with \(\rho = 1\).
Theorem

Let $C$ be a $(w - 1) - (n, w, 1)$-design. Then code

$\tilde{C} = \{x : x \subset \{1, \ldots, n\}, |x| = w + 1, \exists y \in C : y \subset x\}$ is completely regular in $J(n, w + 1)$ with $\rho = 1$. 
Eigenvector of a graph

Let $G$ be a graph. Define the adjacency matrix of graph $G$ as matrix $M$:

$M_{xy} = 1$, if $(x, y) \in E$,

$M_{xy} = 0$, otherwise.

Eigenvector $u$ of graph $G$ is an eigenvector of adjacency matrix of $G$. 
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\textit{Eigenvector $u$ of graph $G$} is an eigenvector of adjacency matrix of $G$. 
Let $u$ be an eigenvector of $J(n, w)$. Define the vector $\hat{u}$, such that for any vertex $x$ of graph $J(n, w')$, $w < w'$

$$\hat{u}_x := \sum_{y \subseteq x} u_y$$

Theorem, Godsil, ”Association schemes”
If $u$ is eigenvector of $J(n, w)$ then $\hat{u}$ is eigenvector of $J(n, w')$. 
Eigenvectors of Johnson graphs

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If $u$ is eigenvector of $J(n, w)$ then $\hat{u}$ is eigenvector of $J(n, w')$. 
Eigenvectors of graphs and completely regular codes with covering radius 1

Lemma (Folklore)

Any completely regular code in $G$ with covering radius 1 is eigenvector of graph $G$, which coordinates takes two different values per se.
The only completely regular codes with $\rho = 1$ to exist in $J(9, 4)$ are codes with the following intersection arrays:

\begin{itemize}
  \item $\{d_1^- = 4, d_0^+ = 5\}$, Code is $\{x : i \in x\}, i \in \{1, \ldots, 9\}$
  \item $\{d_1^- = 15, d_0^+ = 6\}$, “Sporadic” code,
  \item $\{d_1^- = 12, d_0^+ = 9\}$, Code from STS(9).
\end{itemize}
Alltop’s extension constructions

Let $C$ be a $t - (2w + 1, w, \lambda)$-design.

$$C' = \{x \cup 2w + 2 : x \in C\},$$

$$C'' = \{\{1, \ldots, 2w + 1\} \setminus x : x \in C\},$$

**Theorem (Alltop, 1975)**

Let $C$ be a $t - (2w + 1, w, \lambda)$-design with $t \equiv 0 (mod 2)$. Then $C' \cup C''$ is a $t + 1 - (2w + 2, w + 1, \lambda)$-design.

**Proposition**

Let $C$ be a completely regular code in $J(2w + 1, w)$ with $\rho = 1$. Then code $C' \cup C''$ is completely regular in $J(2w + 2, w + 1)$ with $\rho = 1$. 

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\overline{C} = \{ x \subset \{1, \ldots, 2w + 1\} : |x| = w, x \notin C \}.
\]

**Theorem (Alltop, 1975)**

Let \( C \) be a \( t-(2w+1, w, \lambda) \)-design with \( t \equiv 1 (mod 2) \), \( |C| = \binom{2w+1}{w}/2 \). Then \( C' \cup \overline{C}'' \) is a \( t+1-(2w, w, \lambda) \)-design.

**Proposition**

Let \( C \) be a completely regular code in \( J(2w+1, w) \) with \( \rho = 1 \) such that \( |C| = \binom{2w+1}{w}/2 \). Then code \( C' \cup \overline{C}'' \) is completely regular in \( J(2w+2, w+1) \) with \( \rho = 1 \).
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**Proposition**

*Let $C$ be a completely regular code in $J(2w + 1, w)$ with $\rho = 1$ such that $|C| = (2w+1)/2$. Then code $C' \cup \overline{C''}$ is completely regular in $J(2w + 2, w + 1)$ with $\rho = 1$.*
Extension of completely regular codes in $J(9, 4)$

Completely regular code in $J(9, 4)$ with intersection array
\( \{d_1^- = 15, d_0^+ = 6\} \) is extended to completely regular code in $J(10, 5)$ with intersection array \( \{d_1^- = 20, d_0^+ = 8\} \).

Completely regular code in $J(9, 4)$ with intersection array
\( \{d_1^- = 12, d_0^+ = 9\} \) is extended to completely regular code in $J(10, 5)$ with intersection array \( \{d_1^- = 16, d_0^+ = 12\} \).
Extension of completely regular codes in $J(9, 4)$

Completely regular code in $J(9, 4)$ with intersection array
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Studied completely regular codes with $\rho = 1$ in $J(2w + 1, w)$

Enumerated intersection arrays of completely regular codes in Johnson graph $J(9, 4)$ with $\rho = 1$

New construction of completely regular codes from $(w - 1) - (n, w, 1)$-designs

Alltop’s extension constructions applied to completely regular codes in $J(2w + 1, w)$ with $\rho = 1$ give completely regular codes with $\rho = 1$ in $J(2w + 2, w + 1)$
Conclusion

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Conclusion

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**Enumerated intersection arrays of completely regular codes in**
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**New construction of completely regular codes from**
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Thank you for your attention