On completely regular codes in Johnson graphs J(2w+1,w) with covering radius 1

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Code in graph

Code C in a graph G is a collection of vertices of G.

Distance d(x,y) between two vertices x, y is the number of edges is the shortest path, connecting x and y.

Covering radius ρ of code C in graph G is a maximum distance from a vertex of graph to the code C:

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Completely regular code

$C_i = \{x \in V(G) : d(x, C) = i\}, 0 \leq i \leq \rho.$

For x from C_i denote with $d_i^+(x), d_i^0(x), d_i^-(x)$ the number of vertices from C_{i+1}, C_i and C_{i-1} that are adjacent with x.

A code C is called *completely regular*, if for any fixed $i, 0 \leq i \leq \rho(C)$ the numbers $d_i^+(x), d_i^0(x), d_i^-(x)$ does not depend on choice of x from C_i.

Intersection array of completely regular code C:
$\{d_1^-, \ldots, d_\rho^-, d_0^+, \ldots, d_{\rho-1}^+\}$.

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Johnson and Kneser graphs

Johnson graph J(n,w)

$V = \{x \subset \{1, \dots, n\} : |x| = w\}.$
$E = \{(x, y) : |x \cap y| = w - 1\}$.

Kneser graph K(n,w)

$V = \{x \subset \{1, \dots, n\} : |x| = w\}.$
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Subject of inquiry

Completely regular codes in $J(2w+1, w)$ with covering radius 1

Completely regular codes in $J(2w+1, w)$ and $K(2w+1, w)$

From work by Neumaier [1] we get:

Statement

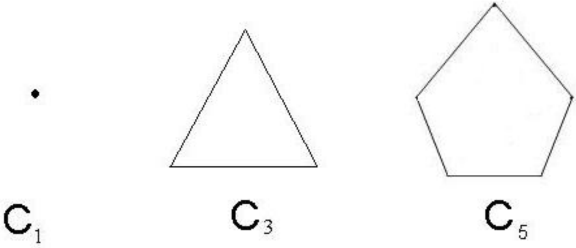
A code C in $J(2w+1, w)$ with $\rho = 1$ is completely regular iff C is completely regular code with $\rho = 1$ in $K(2w+1, w)$.

[1] Neumaier A. Completely regular codes. Discrete Mathematics. 1992. V. 106/107. P. 353-360.

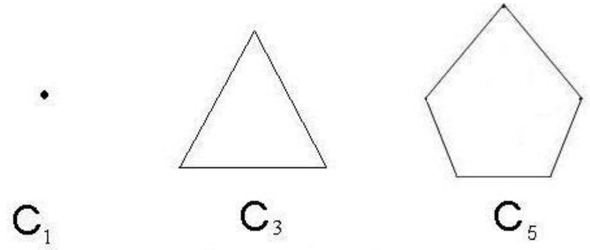
Completely regular code in $J(9,4)$ with array $\{d_1^- = 15, d_0^+ = 6\}$

Completely regular code in $J(9,4)$ with array $\{d_1^- = 15, d_0^+ = 6\}$ exists iff exists completely regular code in $K(9,4)$ with array $\{d_1^- = 5, d_0^+ = 2\}$.

CRC in $K(9,4)$ with array $\{d_1^- = 5, d_0^+ = 2\}$



CRC in $K(9,4)$ with array $\{d_1^- = 5, d_0^+ = 2\}$

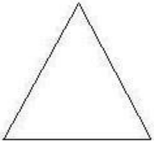


"Orbits":

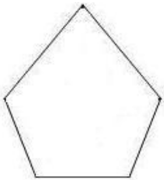
	C_1	C_3	C_5
1	0	0	4
2	1	0	3
3	0	1	$3'$
4	1	1	$2c$
5	0	2	$2c$
6	0	2	$2'$
7	1	2	1
8	1	3	0
9	0	3	1
10	0	1	$3c$
11	1	1	$2'$

CRC in $K(9,4)$ with array $\{d_1^- = 5, d_0^+ = 2\}$

C_1



C_3



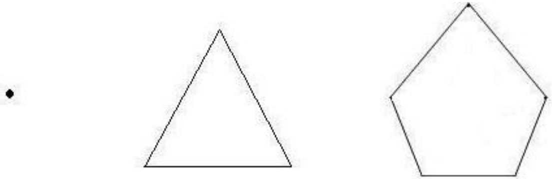
C_5

"Orbits":

	C_1	C_3	C_5
1	0	0	4
2	1	0	3
3	0	1	$3'$
4	1	1	$2c$
5	0	2	$2c$
6	0	2	$2'$
7	1	2	1
8	1	3	0
9	0	3	1
10	0	1	$3c$
11	1	1	$2'$

	1	2	3	4	5	6	7	8	9	10	11
1	*	*	*	*	*	*	*	1	1	0	0
2	*	*	*	*	*	*	*	0	2	0	0
3	*	*	*	*	*	*	*	0	0	0	2
4	*	*	*	*	*	*	*	0	0	2	0
5	*	*	*	*	*	*	*	0	0	1	1
6	*	*	*	*	*	*	*	0	0	0	2
7	*	*	*	*	*	*	*	0	0	2	0
8	*	*	*	*	*	*	*	0	0	0	0
9	*	*	*	*	*	*	*	0	0	0	0
10	*	*	*	*	*	*	*	0	0	0	0
11	*	*	*	*	*	*	*	0	0	0	0

CRC in $K(9,4)$ with array $\{d_1^- = 5, d_0^+ = 2\}$



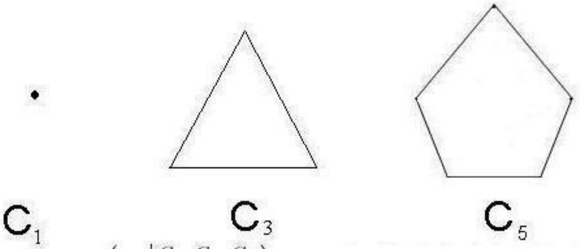
C_1 C_3 C_5

Code

	C_1	C_3	C_5
1	0	0	4
2	1	0	3
3	0	1	3′
4	1	1	2c
5	0	2	2c
6	0	2	2′
7	1	2	1
8	1	3	0
9	0	3	1
10	0	1	3c
11	1	1	2′

	1	2	3	4	5	6	7	8	9	10	11
1	*	*	*	*	*	*	*	1	1	0	0
2	*	*	*	*	*	*	*	0	2	0	0
3	*	*	*	*	*	*	*	0	0	0	2
4	*	*	*	*	*	*	*	0	0	2	0
5	*	*	*	*	*	*	*	0	0	1	1
6	*	*	*	*	*	*	*	0	0	0	2
7	*	*	*	*	*	*	*	0	0	2	0
8	*	*	*	*	*	*	*	0	0	0	0
9	*	*	*	*	*	*	*	0	0	0	0
10	*	*	*	*	*	*	*	0	0	0	0
11	*	*	*	*	*	*	*	0	0	0	0

CRC in $K(9,4)$ with array $\{d_1^- = 5, d_0^+ = 2\}$



Code (rows 1–7)

	C_1	C_3	C_5
1	0	0	4
2	1	0	3
3	0	1	$3'$
4	1	1	$2c$
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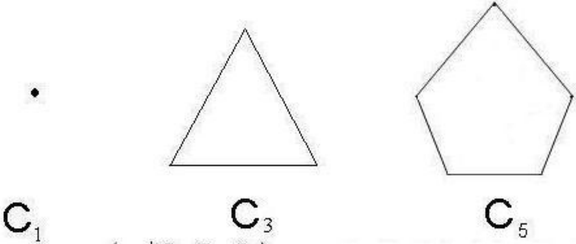
Completely regular code in Kneser graph K(9,4) with intersection array:

$$\{d_1^- = 5 \;,\; d_0^+ = 2\}$$

CRC in $K(9,4)$ with array $\{d_1^- = 5, d_0^+ = 2\}$



Code

	C_1	C_3	C_5
1	0	0	4
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7	1	2	1
8	1	3	0
9	0	3	1
10	0	1	$3c$
11	1	1	$2'$

Completely regular code in Johnson graph J(9,4) with intersection array:

$$\{d_1^- = 15,\ d_0^+ = 6\}$$

Completely regular codes from $(w-1)-(n, w, 1)$-designs

Theorem (Martin '98)

Any simple $(w-1)-(n, w, \lambda)$-design is completely regular in $J(n, w)$ with $\rho = 1$.

Theorem

Let C be a $(w-1)-(n,w,1)$-design. Then code $\widetilde{C} = \{x : x \subset \{1,\ldots,n\}, |x| = w+1, \exists y \in C : y \subset x\}$ is completely regular in $J(n, w+1)$ with $\rho = 1$.

Eigenvector of a graph

Let G be a graph. Define *adjacency matrix* of graph G as matrix M:
$M_{xy} = 1$, if $(x, y) \in E$,
$M_{xy} = 0$, otherwise.

Eigenvector u of graph G is an eigenvector of adjacency matrix of G.

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Eigenvectors of Johnson graphs

Let u be an eigenvector of $J(n, w)$.
Define the vector $\widehat{u}$, such that for any vertex x of graph $J(n, w'), w < w'$

$$\widehat{u}_x := \sum_{y \subset x} u_y$$

Theorem, Godsil, "Association schemes"

If u is eigenvector of $J(n, w)$ then $\widehat{u}$ is eigenvector of $J(n, w')$.

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Eigenvectors of graphs and completely regular codes with covering radius 1

Lemma (Folklore)

Any completely regular code in G with covering radius 1 is eigenvector of graph G, which coordinates takes two different values per se.

Completely regular codes in J(9,4) with $\rho = 1$

Theorem

The only completely regular codes with $\rho = 1$ to exist in $J(9,4)$ are codes with the following intersection arrays:
$\{d_1^- = 4, d_0^+ = 5\}$, Code is $\{x : i \in x\}$, $i \in \{1, \ldots, 9\}$
$\{d_1^- = 15, d_0^+ = 6\}$, "Sporadic" code,
$\{d_1^- = 12, d_0^+ = 9\}$, Code from STS(9).

Alltop's extension constructions

Let C be a $t-(2w+1, w, \lambda)$-design.

$$C' = \{x \cup 2w+2 : x \in C\},$$

$$C'' = \{\{1, \ldots, 2w+1\} \setminus x : x \in C\},$$

Theorem (Alltop, 1975)

Let C be a $t-(2w+1, w, \lambda)$-design with $t \equiv 0(mod2)$. Then $C' \cup C''$ is a $t+1-(2w+2, w+1, \lambda)$-design.

Proposition

Let C be a completely regular code in $J(2w+1, w)$ with $\rho = 1$. Then code $C' \cup C''$ is completely regular in $J(2w+2, w+1)$ with $\rho = 1$.

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Theorem (Alltop, 1975)

Let C be a $t-(2w+1, w, \lambda)$-design with $t \equiv 1(mod 2)$, $|C| = \binom{2w+1}{w}/2$. Then $C' \cup \overline{C}''$ is a $t+1-(2w, w, \lambda)$-design.

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Extension of completely regular codes in $J(9,4)$

Completely regular code in $J(9,4)$ with intersection array $\{d_1^- = 15, d_0^+ = 6\}$ is extended to completely regular code in $J(10,5)$ with intersection array $\{d_1^- = 20, d_0^+ = 8\}$.

Completely regular code in $J(9,4)$ with intersection array $\{d_1^- = 12, d_0^+ = 9\}$ is extended to completely regular code in $J(10,5)$ with intersection array $\{d_1^- = 16, d_0^+ = 12\}$.

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Conclusion

Studied completely regular codes with $\rho = 1$ in $J(2w+1, w)$

Enumerated intersection arrays of completely regular codes in Johnson graph $J(9,4)$ with $\rho = 1$

New construction of completely regular codes from $(w-1)-(n, w, 1)$-designs

Alltop's extension constructions applied to completely regular codes in $J(2w+1, w)$ with $\rho = 1$ give completely regular codes with $\rho = 1$ in $J(2w+2, w+1)$

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Thank you for your attention