# Cocharacters of polynomial identities of upper triangular matrices

**Silvia Boumova, Vesselin Drensky**

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# Introduction

- Let $K$ be a field of characteristic 0;
- Consider unital associative algebras over $K$;
- Let $R$ be a PI-algebra;
- Let $T(R) \subset K\langle X\rangle = K\langle x_1, x_2, \ldots\rangle$ be the T-ideal of its polynomial identities, where $K\langle X\rangle$ is the free associative algebra of countable rank;

- One of the most important objects in the quantitative study of the polynomial identities of $R$ is the cocharacter sequence

$$\chi_n(R) = \sum_{\lambda \vdash n} m_\lambda(R)\chi_\lambda, \quad n = 0, 1, 2, \ldots,$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a partition of $n$ and $\chi_\lambda$ is the corresponding irreducible character of the symmetric group $S_n$;

- The $n$-th cocharacter

$$\chi_n(R) = \chi_{S_n}(P_n/(T(R) \cap P_n))$$

is equal to the character of the representation of $S_n$ acting on the vector subspace $P_n \subset K\langle X \rangle$ of the multilinear polynomials of degree $n$ modulo the polynomial identities of $R$.

**Main problem:**

Find the cocharacter sequence $\chi_n(R)$ for a given algebra $R$.

# History

The explicit form of the multiplicities $m_\lambda(R)$ is known for few algebras only

- **(1973, Krakowski and Regev)** The Grassmann algebra $E$;
- **(1984, Formanek and Drensky)** The $2 \times 2$ matrix algebra $M_2(K)$;
- **(Folklorely known, e.g. 1999, Mishchenko, Regev, Zaicev)** The algebra $U_2(K)$ of the $2 \times 2$ upper triangular matrices;
- **(1982, Popov and 1991 Carini, Di Vincenzo)** The tensor square $E \otimes E$ of the Grassmann algebra.

- The cocharacter sequences are related with another important group action, namely the action of the general linear group $GL_d = GL_d(K)$ on the $d$-generated free subalgebra $K\langle x_1, \dots, x_d\rangle \subset K\langle X\rangle$ modulo the polynomial identities in $d$ variables of $R$.
- The algebra

$$F_d(R) = K\langle x_1, \dots, x_d\rangle / (K\langle x_1, \dots, x_d\rangle \cap T(R))$$

is called **the relatively free algebra of rank** $d$ in the variety of algebras $\mathrm{var}(R)$ generated by the algebra $R$.

- The algebra $F_d(R)$ is $\mathbb{Z}_d$-graded with grading defined by:

$$\begin{array}{c} \deg(x_1) = (1, 0, \dots, 0), \\ \deg(x_2) = (0, 1, \dots, 0), \\ \dots \\ \deg(x_d) = (0, 0, \dots, 1). \end{array}$$

- The Hilbert series

$$\begin{aligned} H(F_d(R), T_d) &= H(F_d(R), t_1, \ldots, t_d) \\ &= \sum_{n_i \geq 0} \dim(F_d^{(n_1,\ldots,n_d)}(R)) t_1^{n_1} \cdots t_d^{n_d} \end{aligned}$$

of $F_d(R)$, where $F_d^{(n_1,\ldots,n_d)}(R)$ is the homogeneous component of degree $(n_1, \ldots, n_d)$ of $F_d(R)$, is a symmetric function which plays the role of the character of the corresponding $GL_d$-representation.

## Schur functions

$$S_\lambda(X) = \frac{V(\lambda+\delta, X)}{V(\lambda, X)},$$

where $\lambda = (\lambda_1, \ldots, \lambda_d)$, $\delta = (d-1, d-2, \ldots, 2, 1, 0)$ and $\mu = (\mu_1, \ldots, \mu_d)$

$$V(\mu, X) = \begin{vmatrix} x_1^{\mu_1} & x_2^{\mu_1} & \ldots & x_d^{\mu_1} \\ x_1^{\mu_2} & x_2^{\mu_2} & \ldots & x_d^{\mu_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\mu_d} & x_2^{\mu_d} & \ldots & x_d^{\mu_d} \end{vmatrix}$$

- The Schur functions $S_\lambda(T_d) = S_\lambda(t_1, \dots, t_d)$ are the characters of the irreducible components of $F_d(R)$ and

$$H(F_d(R), T_d) = \sum_\lambda m_\lambda(R) S_\lambda(T_d), \quad \lambda = (\lambda_1, \dots, \lambda_d).$$

- **(1982, Berele and 1981, 1984, Drensky)** The multiplicities $m_\lambda(R)$ are the same as in the cocharacter sequence $\chi_n(R)$, $n = 0, 1, 2, \dots$.
- Hence, if we know the Hilbert series $H(F_d(R), T_d)$, we can find the multiplicities $m_\lambda(R)$ in $\chi_n(R)$ for those $\lambda$ which are partitions in not more than $d$ parts.

- Following the idea of **Drensky and Genov** (2003) we consider the multiplicity series of $R$

$$\begin{aligned} M(R; T_d) &= M(R, t_1, \dots, t_d) = \sum_\lambda m_\lambda(R) T_d^\lambda \\ &= \sum_\lambda m_\lambda(R) t_1^{\lambda_1} \cdots t_d^{\lambda_d}. \end{aligned}$$

- This is the generating function of the cocharacter sequence of $R$ which corresponds the multiplicities $m_\lambda(R)$ when $\lambda$ is a partition in $\leq d$ parts.

- It is also convenient to consider the subalgebra $\mathbb{C}[[V_d]] \subset \mathbb{C}[[T_d]]$ of the formal power series in the new set of variables $V_d = \{v_1, \dots, v_d\}$, where

$$v_1 = t_1, v_2 = t_1 t_2, \dots, v_d = t_1 \cdots t_d.$$

- Then the multiplicity series $M(f; T_d)$ can be written as

$$M'(f; V_d) = \sum_{\lambda} m_\lambda v_1^{\lambda_1 - \lambda_2} \cdots v_{d-1}^{\lambda_{d-1} - \lambda_d} v_d^{\lambda_d} \in \mathbb{C}[[V_d]].$$

  We call $M'(f; V_d)$ also the multiplicity series of $f$. The advantage of the mapping $M' : \mathbb{C}[[T_d]]^{S_d} \to \mathbb{C}[[V_d]]$ defined by $M' : f(T_d) \to M'(f; V_d)$ is that it is a bijection.

- For a PI-algebra $R$ we define the multiplicity series of $R$

$$\begin{aligned} M(R;T_d) &= M(R,t_1,\ldots,t_d) = \textstyle\sum_{\lambda} m_{\lambda}(R)T_d^{\lambda} \\ &= \textstyle\sum_{\lambda} m_{\lambda}(R)t_1^{\lambda_1}\cdots t_d^{\lambda_d}. \end{aligned}$$

- Similarly we define the series $M'(R;V_d)$.

- Then, if we know the Hilbert series $H(F_d(R), T_d)$, the problem is to compute the multiplicity series $M(R; T_d)$ and to find its coefficients.
- This problem was solved by **Drensky and Genov** (2004) for rational symmetric functions of special kind and in two variables.
- **Berele** (2006) suggested another approach involving the so called nice rational functions which allowed (**2008, Berele, Regev**) to solve for unital algebras the conjecture of Regev about precise asymtotics of the growth of the codimention sequence of PI-algebras.
- But the results of **Berele** and **Berele, Regev** do not give explicit algorithms to find the multiplicities of the irreducible characters.

- In the present paper we study the cocharacter sequence of the algebra $U_k = U_k(K)$ of $k \times k$ upper triangular matrices.
- The algebra $U_k$ is one of the central objects in the theory of PI-algebras satisfying a nonmatrix polynomial identity (i.e., an identity which does not hold for the $2 \times 2$ matrix algebra $M_2(K)$).
- **Latyshev** (1966) proved that every finitely generated PI-algebra with a nonmatrix identity satisfies the identities of $U_k$ for a suitable $k$.

- **Yu. Maltsev** (1971) showed that the polynomial identities of $U_k$ follow from the identity

$$[x_1, x_2] \cdots [x_{2k-1}, x_{2k}] = 0,$$

where $[x, y] = xy - yx$ is the commutator of $x$ and $y$.

- This means that $T(U_k) = C^k$, where

$$C = T(K) = K\langle X\rangle[K\langle X\rangle, K\langle X\rangle]K\langle X\rangle$$

is the commutator ideal of $K\langle X\rangle$.

# Preliminaries

- Every symmetric function $f(T_d) \in \mathbb{C}[[T_d]]^{S_d}$ can be presented in the form

$$f(T_d) = \sum_{\lambda} m_\lambda S_\lambda(T_d), \quad m_\lambda \in \mathbb{C}, \lambda = (\lambda_1, \dots, \lambda_d),$$

where $S_\lambda(T_d)$ is the Schur function related to $\lambda$.

- We associate with $f(T_d)$ its multiplicity series

$$M(f; T_d) = \sum_{\lambda} m_\lambda T_d^\lambda = \sum_{\lambda} m_\lambda t_1^{\lambda_1} \cdots t_d^{\lambda_d} \in \mathbb{C}[[T_d]].$$

## Berele, 2006

*The functions* $f(T_d)\in\mathbb{C}[[T_d]]^{S_d}$ *and* $M(f;T_d)$ *are related by the following equality. If*

$$f(T_d)\prod_{i<j}(t_i-t_j)=\sum_{p_i\geq 0}b(p_1,\ldots,p_d)t_1^{p_1}\cdots t_d^{p_d},\quad b(p_1,\ldots,p_d)\in\mathbb{C},$$

*then*

$$M(f;T_d)=\frac{1}{t_1^{d-1}\cdots t_{d-2}^2t_{d-1}}\sum_{p_i>p_{i+1}}b(p_1,\ldots,p_d)t_1^{p_1}\cdots t_d^{p_d},$$

*where the summation is on all* $p=(p_1,\ldots,p_d)$ *such that* $p_1>p_2>\cdots>p_d$.

## Remark

In the general case, it is difficult to find $M(f;T_d)$ if we know $f(T_d)$. But it is very easy to check whether the formal power series

$$h(T_d)=\sum h(q_1,\dots,q_d)t_1^{q_1}\cdots t_d^{q_d},\quad q_1\geq\cdots\geq q_d,$$

is equal to the multiplicity series $M(f;T_d)$ of $f(T_d)$ because $h(T_d)=M(f;T_d)$ if and only if

$$f(T_d)\prod_{i<j}(t_i-t_j)=\sum_{\sigma\in S_d}\mathrm{sign}(\sigma)t_{\sigma(1)}^{d-1}t_{\sigma(2)}^{d-2}\cdots t_{\sigma(d-1)}h(t_{\sigma(1)},\dots,t_{\sigma(d)}).$$

This equation can be used to verify the computational results on multiplicities.

- If two symmetric functions $f(T_d)$ and $g(T_d)$ are related by

$$f(T_d) = g(T_d) \prod_{i=1}^{d} \frac{1}{1 - t_i},$$

then $f(T_d)$ is Young derived from $g(T_d)$ and the decomposition of $f(T_d)$ as a series of Schur functions can be obtained from the decomposition of $g(T_d)$ using the Young rule.

**Preposition** (Drensky and Genov, 2003)

Let $Y$ be the linear operator in $\mathbb{C}[[V_d]] \subset \mathbb{C}[[T_d]]$ which sends the multiplicity series of the symmetric function $g(T_d)$ to the multiplicity series of its Young derived $f(T_d)$. Then

$$Y(M(g);T_d) = M(f;T_d) = M\left(g(T_d)\prod_{i=1}^{d}\frac{1}{1-t_i};T_d\right)$$

$$=\prod_{i=1}^{d}\frac{1}{1-t_i}\sum(-t_2)^{\varepsilon_2}\ldots(-t_d)^{\varepsilon_d}M(g;t_1t_2^{\varepsilon_2},t_2^{1-\varepsilon_2}t_3^{\varepsilon_3}\ldots t_{d-1}^{1-\varepsilon_{d-1}}t_d^{\varepsilon_d},t_d^{1-\varepsilon_d}),$$

where the summation runs on all $\varepsilon_2,\ldots,\varepsilon_d = 0,1$.

# Main results

## Theorem

*The Hilbert series $H(F_d(U_k), T_d)$ of the algebra $F_d(U_k)$ is*

$$H(F_d(U_k), T_d) = \frac{1}{t_1+\cdots+t_d-1}\left(\left(1+(t_1+\cdots+t_d-1)\prod_{i=1}^{d}\frac{1}{1-t_i}\right)^k - 1\right)$$

$$= \sum_{j=1}^{k}\binom{k}{j}\left(\prod_{i=1}^{d}\frac{1}{1-t_i}\right)^j (t_1+\cdots+t_d-1)^{j-1}.$$

- For the proof we use a result of **Formanek** (1985) for the Hilbert series $H(K\langle x_1,\dots,x_d\rangle/T(R_1)T(R_2))$.

- Using the decomposition

$$(t_1+\cdots+t_d)^q=\sum_{\lambda\vdash q} d_\lambda S_\lambda(T_d),$$

where $d_\lambda$ is the degree of the irreducible $S_q$-character $\chi_\lambda$, it is sufficient to apply the Young rule up to $k$ times on the Schur functions $S_\lambda(T_d)$ for all partitions $\lambda$ of $q\leq k-1$.

## Corollary

*Let $Y$ be the linear operator in $\mathbb{C}[[V_d]] \subset \mathbb{C}[[T_d]]$ which sends the multiplicity series of the symmetric function $g(T_d)$ to the multiplicity series of its Young derived:*

$$Y(M(g); T_d) = M\left(g(T_d)\prod_{i=1}^{d}\frac{1}{1-t_i}; T_d\right).$$

*Then the multiplicity series of $U_k$ is*

$$M(U_k; T_d) = \sum_{j=1}^{k}\sum_{q=0}^{j-1}\sum_{\lambda\vdash q}(-1)^{j-q-1}\binom{k}{j}\binom{j-1}{q}d_\lambda Y^j(T_d^\lambda),$$

*where $d_\lambda$ is the degree of the irreducible $S_n$-character $\chi_\lambda$ and $T_d^\lambda = t_1^{\lambda_1}\cdots t_d^{\lambda_d}$ for $\lambda = (\lambda_1, \ldots, \lambda_d)$.*

## Theorem

*The multiplicity series and the multiplicities of the cocharacter sequence of the algebra $U_k$ of the $k \times k$ upper triangular matrices for $k = 1, 2$ are*

$$M'(U_1; V) = \frac{1}{1 - v_1}, \quad m_\lambda(U_1) = \begin{cases} 1, & \lambda = (\lambda_1) \\ 0, & \lambda_2 > 0; \end{cases}$$

$$M'(U_2; V) - M'(U_1; V) = \frac{v_2 + v_3}{(1 - v_1)^2(1 - v_2)},$$

$$m_\lambda(U_2) - m_\lambda(U_1) = \begin{cases} \lambda_1 - \lambda_2 + 1, & \lambda = (\lambda_1, \lambda_2), \lambda_2 > 0, \\ \lambda_1 - \lambda_2 + 1, & \lambda = (\lambda_1, \lambda_2, 1), \\ 0, & \text{for all other } \lambda. \end{cases}$$

## Theorem

(i) *The difference of the multiplicity series of $U_3$ and $U_2$ is*

$$M'(U_3;V)-M'(U_2;V)=$$

$$\left(\frac{v_5+v_4^2+4v_4+4v_3}{1-v_3}+v_2^2\right)\frac{1-v_1v_2}{(1-v_1)^3(1-v_2)^3}-\frac{(v_2^2-v_1-3v_2+3)v_4+(v_1v_2^2-v_1v_2+v_2^2-v_1-4v_2+4)v_3}{(1-v_1)^3(1-v_2)^3},$$

(ii) *The explicit form of the corresponding multiplicities is*

$$m_\lambda(U_3)-m_\lambda(U_2)=\begin{cases} n_\lambda, & \lambda=(\lambda_1,\lambda_2,\lambda_3,1,1),\\ n_\lambda, & \lambda=(\lambda_1,\lambda_2,\lambda_3,2),\\ 4n_\lambda-c_\lambda, & \lambda=(\lambda_1,\lambda_2,\lambda_3,1),\\ 4n_\lambda-c_\lambda, & \lambda=(\lambda_1,\lambda_2,\lambda_3),\lambda_3>0,\\ \frac{1}{2}\lambda_1(\lambda_1-\lambda_2+1)(\lambda_2-1), & \lambda_2\geq 2,\\ 0, & \textit{for all other } \lambda,\end{cases}$$

## Theorem

*where*

$$n_\lambda = \frac{1}{2}(\lambda_1 - \lambda_2 + 1)(\lambda_2 - \lambda_3 + 1)(\lambda_1 - \lambda_3 + 2).$$

*and the correction $c_\lambda$ is*

$$c_\lambda = \begin{cases} \frac{1}{2}(\lambda_1 + 2)(\lambda_1 - \lambda_2 + 1)(\lambda_2 + 1), & \lambda = (\lambda_1, \lambda_2, 1, 1), \\ \frac{1}{2}(\lambda_1 + 3)(\lambda_1 - \lambda_2 + 1)(\lambda_2 + 2), & \lambda = (\lambda_1, \lambda_2, 1), \\ 0, & \textit{for all other } \lambda. \end{cases}$$

## Theorem

(i) *If* $m_\lambda(U_k) \neq 0$, *then* $\lambda = (\lambda_1, \ldots, \lambda_{2k-1})$ *is a partition in not more than* $2k-1$ *parts and* $\overline{\lambda} = (\lambda_{k+1}, \ldots, \lambda_{2k-1})$ *is a partition of* $i \leq k-1$.

(ii) *If* $\overline{\lambda}$ *is a partition of* $n-1$, *then* $m_\lambda(U_k) = d_{\overline{\lambda}} n_\mu$, *where* $d_{\overline{\lambda}}$ *is the degree of the* $S_{k-1}$*-character* $\chi_{\overline{\lambda}}$,

$$\mu = (\lambda_1 - \lambda_{k+1}, \ldots, \lambda_k - \lambda_{k+1}),$$

$$M\left(\prod_{i=1}^{k} \frac{1}{(1-t_i)^k}\right) = \sum_{\mu} n_\mu T_k^\mu,$$

*i.e.,* $n_\mu = \prod_{1 \leq i < j \leq k} \frac{\lambda_i - \lambda_j + j - i}{j - i}$ *is the coefficient of* $S_\mu(T_k)$ *in the decomposition of* $\prod 1/(1-t_i)^k$.