Cryptanalysis of Block Ciphers via Decoding of Long Reed-Muller Codes

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Introduction

Reconstruction of the key

Decoding of RM codes with repeated symbols

Introduction

- Linear Cryptanalysis: powerful tool in the analysis of block ciphers (Matsui, 1993)
- *Step 1:* finding **linear relations** involving **key**, **plaintext** and **ciphertext** bits with probability $1/2 + \varepsilon$
- *Step 2:* using these relations and a sample of plaintext-ciphertext pairs to recover some key bits.
- *data complexity*: $\mathcal{O}(1/\varepsilon^2)$
- reducing this data-complexity:
  - multiple linear relations: Kaliski-Robshaw (1994), Biryukov-De Cannière-Quisquater (2004), Gérard-Tillich (2007), Fourquet-Loidreau-Tavernier (2009)
  - non-linear relations: Knudsen-Robshaw (1996), Shimoyama-Kaneko (1998), Tokareva (2008)

Proposed Algorithm: given non-linear (low order) relations between plaintext, ciphertext and key bits, reconstruct the key bits (Step 2).

- this reconstruction is translated to a soft decoding problem in the Reed-Muller (RM) codes, with *repeated symbols* .
- these codes can be decoded with quasi-linear complexity
- it is a general purpose algorithm
- it is hard to find non-linear approximations satisfying the error-probability threshold allowed by this algorithm.
- any decoding algorithm may be used (e.g. in RM codes of order 2)

Block Ciphers

- A block cipher is a vectorial Boolean function:

$$\begin{array}{rcccccc} E: & \mathbb{F}_2^u & \times & \mathbb{F}_2^v & \longrightarrow & \mathbb{F}_2^w \\ & (X & , & K) & \longmapsto & Y = E(X,K) \end{array}$$

  $X = (X_1, \ldots, X_u)$: plaintext
  $K = (K_1, \ldots, K_v)$: key
  $Y = (Y_1, \ldots, Y_w)$: ciphertext

- A relation between X, K and $Y = E(X,K)$ with bias $0 < \varepsilon \leq 1/2$ is given by a Boolean function $F : \mathbb{F}_2^{u+v+w} \to \mathbb{F}_2$ such that:

$$F(X,K,Y) = 0 \text{ with probability } 1/2 + \varepsilon$$

- *Linear* cryptanalysis: F is a linear function
- for clarity, we consider $F : \mathbb{F}_2^{u+v} \to \mathbb{F}_2$ and $G : \mathbb{F}_2^w \to \mathbb{F}_2$ such that

$$F(X,K) = G(E(X,K)) \text{ with probability } 1/2 + \varepsilon$$

Reconstruction of the key

Notation: for $i = (i_1, \ldots, i_u) \in \mathbb{F}_2^u$ and $j = (j_1, \ldots, j_v) \in \mathbb{F}_2^v$:

$$X^i := X_1^{i_1} X_2^{i_2} \cdots X_u^{i_u} \quad \text{and} \quad K^j := K_1^{j_1} K_2^{j_2} \cdots K_v^{j_v}$$

Let F have the following polynomial representation:

$$\begin{aligned} F(X, K) &= \sum_{i,j} a_{i,j} X^i K^j \in \mathbb{F}_2[X, K] \\ F_K(X) &= \sum_i (\sum_j a_{i,j} K^j) X^i \in \mathbb{F}_2[X] \quad \text{when K is fixed} \end{aligned}$$

with $a_{i,j} \in \mathbb{F}_2$.

Using a sample of plaintext-ciphertext pairs associated with a fixed key $\bar{K}$, we will reconstruct the polynomial $F_{\bar{K}}(X) = F(X, \bar{K})$

$\longrightarrow$ this gives the coefficients of the form $\sum_j a_{i,j} K^j$.

RM codes with repeated symbols

For complexity reasons, we will assume that:

- $F_K(X)$ depends on a small number $u' \ll u$ of variables, say $X_1, \ldots, X_{u'}$.
  For $X \in \mathbb{F}_2^u$, let $X' = (X_1, \ldots, X_{u'})$.
- The degree h of $F_K(X)$ is small.

$$F_K(X) = F_K(X') \in \mathrm{RM}(h, u').$$

Moreover, we assume that, when K is fixed, the relation between $F_K(X)$ and $E_K(X) = E(X, K)$ still holds:

$$\Pr_X[G(E_K(X)) = F_K(X')] = \frac{1}{2} + \varepsilon$$

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Let S be a sample of size L of plaintext-ciphertext pairs $(X, Y = E_K(X))$ associated with the key K.

For $x \in \mathbb{F}_2^{u'}$, let

$$S_x = \{(X, Y) \in S | X' = x\},$$

of size L_x ($\sum_{x \in \mathbb{F}_2^{u'}} L_x = L$).

- *$F_K(X') \in \mathrm{RM}(h, u')$ is transmitted L_x times at position $x \in \mathbb{F}_2^{u'}$ over a channel with error probability $1/2 - \varepsilon$,*

with received symbols $(G(Y))_{(X,Y) \in S_x}$:

$$\text{position } x \text{ of } F_K(X') \overset{p=1/2-\varepsilon}{\longrightarrow} (G(Y))_{(X,Y) \in S_x}$$

Recovering the key bits

Let s_x be the Hamming weight of $(G(Y))_{(X,Y)\in S_x}$. We form the vector:

$$(s_1, s_2, \ldots, s_{2^{u'}})$$

We construct the received vector y of length $2^{u'}$ with "hard decoding": at position x, we set

$$y_x = \begin{cases} 0 & \text{if } s_x < L_x/2 \\ 1 & \text{otherwise} \end{cases}$$

Then, the vector y is decoded into the codeword $F_K(X') \in \mathrm{RM}(h, u')$ $\longrightarrow$ we obtain the coefficients $\sum_j a_{i,j} K^j$.

An example

Let

$$F(X,K) = X_1K_2 + X_2K_4 + X_3K_1K_5 + K_1$$

be an approximation of $G(E(X,K))$, with $X, K, Y = E(X,Y) \in \mathbb{F}_2^{64}$.
Here $u' = 3$, $X' = (X1, X2, X3)$, $\deg(F_K(X)) = 1 \Rightarrow F_K(X) \in \mathrm{RM}(1,3)$.
Using a sample S of plaintext-ciphertext pairs, we construct the "received" vector y as above:

$$y = (y_1, \ldots, y_8)$$

To reconstruct $F_K(X)$, we decode y into the nearest affine function

$$A(X') = a_0 + a_1X_1 + a_2X_2 + a_3X_3 \in RM(1,3)$$

which maximizes the quantity $\sum_x (-1)^{y(x)+A(x)}$ (FFT).
Then we obtain:

$$K_1 = a_0, K_2 = a_1, K_4 = a_2 \text{ and } K_1K_5 = a_3.$$

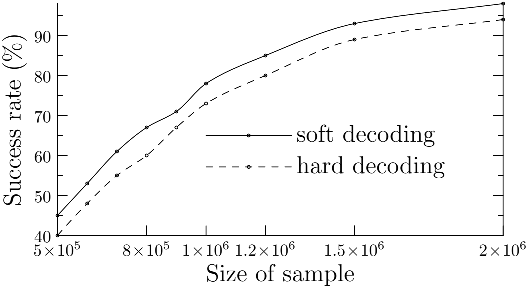
Example of the DES

The DES is a block-cipher with plaintext, ciphertext and key of size 64 bits. We found 20 quadratic approximations of the 8-round DES, with biases $\varepsilon \approx 0.001$. They all imply 6 bits of the key, and are of the form:

$K_9 + K_4K_{13} + K_{15} + K_4K_{15} + K_{13}K_{30} + K_{31} + K_{33} + K_{41} + K_{44} + K_4K_{47} + K_{30}K_{47} + K_{52} + K_{54} + K_{15}K_{54} + K_{47}K_{54} + X_{47} + X_0 + K_{15}X_0 + K_{47}X_0 + X_7 + X_{18} + X_{24} + K_4X_{27} + K_{30}X_{27} + K_{54}X_{27} + X_0X_{27} + X_{28} + K_4X_{28} + K_{54}X_{28} + X_0X_{28} + X_{29} + K_{13}X_{29} + K_{15}X_{29} + K_{47}X_{29} + X_{27}X_{29} + X_{28}X_{29} + K_{13}X_{30} + K_{47}X_{30} + X_{27}X_{30} + K_4X_{31} + K_{30}X_{31} + X_{29}X_{31} + X_{30}X_{31} = Y_{12} + Y_{16} + Y_{39} + Y_{50} + Y_{56}$.

$\longrightarrow$ results similar to using multiple linear relations.

Success rate (all 6 key bits are recovered) of the cryptanalysis:



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Decoding of RM codes with repeated symbols

Recall the results of I. Dumer ("Soft decision decoding of Reed-Muller codes: a simplified algorithm", *IEEE* 2006):

Theorem

Consider long codes $\mathrm{RM}(r,m)$ *such that* $\frac{(m-r)}{\ln m} \to \infty$ *as* $m \to \infty$. *Then these codes can be decoded on a* BSC_p *with complexity of order* $(3n\log_2 n)/2$, *and have a vanishing output block error probability if* $p \leq 1/2 - \varepsilon_1$, *where*

$$\varepsilon_1 = 2\left(\frac{4m}{d}\right)^{1/2^r}. \tag{1}$$

Now, every codeword is transmitted L times over the same binary symmetric channel BSC_p with an error probability $p = 1/2 - \varepsilon$.
In the case of hard decoding, the above theorem gives the following improved threshold:

$$\varepsilon_L = 2\sqrt{\frac{2}{L}}\left(\frac{4m}{d}\right)^{1/2^r}. \tag{2}$$

A soft-decision version

Instead of the hard decoding, we use a soft version approach. Each symbol c_x of a code $\mathrm{RM}(r, m)$ is transmitted L times and is received as some vector g_x of length L and Hamming weight s_x.
For all s, we have:

$$Q(s) := \Pr[s_x = s | c_x = 0] = \binom{L}{s} p^s q^{L-s},$$
$$P(s) := \Pr[s_x = s | c_x = 1] = \binom{L}{s} q^s p^{L-s},$$

Then using the Bayes formula:

$$\Pr[c_x = 0 | s_x] = \frac{Q(s_x)}{P(s_x) + Q(s_x)}$$
$$\Pr[c_x = 1 | s_x] = \frac{P(s_x)}{P(s_x) + Q(s_x)}$$

The received soft-vector $y \in \mathbb{R}^{2^{u'}}$ will be constructed at position x as follows:

$$y_x = \Pr[c_x = 0|s_x] - \Pr[c_x = 1|s_x] = \frac{Q(s_x) - P(s_x)}{P(s_x) + Q(s_x)}$$

With this setting, the decoding threshold is further improved:

Theorem

Consider long codes $\mathrm{RM}(r, m)$ *such that* $\frac{(m-r)}{\ln m} \to \infty$ *as* $m \to \infty$*. Then an* L*-repeated RM code can be decoded on a* BSC_p *with a vanishing output error probability if* $p \leq 1/2 - \varepsilon_L$*, where*

$$\varepsilon_L = \frac{2}{\sqrt{L}} \left(\frac{4m}{d}\right)^{1/2^r} = \frac{\varepsilon_1}{\sqrt{L}}. \tag{3}$$