ON A CLASS OF GRIESMER CODES RELATED TO CAPS

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1. Preliminaries

$\mathbb{F}_q$, $q = p^r$, p – prime, the field with q elements

Definition. A **multiset** in $\mathrm{PG}(k-1, q)$ is a mapping

$$\mathcal{K} : \left\{ \begin{array}{lcl} \mathcal{P} & \rightarrow & \mathbb{N}_0, \\ P & \rightarrow & \mathcal{K}(P). \end{array} \right.$$

$\mathcal{K}(P)$ – the **multiplicity** of the point P.

$\mathcal{Q} \subset \mathcal{P}$: $\mathcal{K}(\mathcal{Q}) = \sum_{P \in \mathcal{Q}} \mathcal{K}(P)$.

$\mathcal{K}(\mathcal{P})$ – the **cardinality** of $\mathcal{K}$.

Points, lines, ... ,hyperplanes of multiplicity i are called i-points, i-lines, ... , i-hyperplanes.

a_i – the number of i-hyperplanes

$(a_i)_{i\geq 0}$ – the **spectrum** of $\mathcal{K}$

Definition. (n, w)-**arc** in $\mathrm{PG}(k-1, q)$: a multiset $\mathcal{K}$ with

1) $\mathcal{K}(\mathcal{P}) = n$;

2) for every hyperplane H: $\mathcal{K}(H) \leq w$;

3) there exists a hyperplane H_0: $\mathcal{K}(H_0) = w$.

Definition. (n, w)-**blocking set with respect to hyperplanes** in $\mathrm{PG}(k-1, q)$ (or (n, w)-**minihyper**): a multiset $\mathcal{K}$ with

1) $\mathcal{K}(\mathcal{P}) = n$;

2) for every hyperplane H: $\mathcal{K}(H) \geq w$;

3) there exists a hyperplane H_0: $\mathcal{K}(H_0) = w$.

2. Arcs and linear codes

Theorem. The existence of an $[n, k, d]_q$-code of full length is equivalent to that of an $(n, n-d)$-arc in $\mathrm{PG}(k-1, q)$.

◇ C – $[n, k, d]_q$-code with $n = t + g_q(k, d)$;

◇ $\mathcal{K}$ - $(n, n-d)$-arc associated with C;

◇ γ_i := maximal multiplicity of an i-dimensional subspace of $\mathrm{PG}(k-1, q)$, $i = 0, 1, \ldots, k-1$,

$$\gamma_i \leq t + g_q(i+1, d).$$

Problem A. For given k, d and q find the smallest value of n for which there exists an $[n, k, d]_q$-code.

The **Griesmer** bound:

$$n_q(k, d) \geq g_q(k, d) = \sum_{i=0}^{k-1} \lceil \frac{d}{q^i} \rceil$$

Problem B. Characterize geometrically all Griesmer codes with given parameters k, d and q. Equivalently: Characterize all minihypers in $\mathrm{PG}(k-1, q)$ with

$$\left(\sum_{i=0}^{k-2} \epsilon_i v_{i+1}, \sum_{i=0}^{k-2} \epsilon_i v_i, \right), 0 \leq \epsilon_i \leq q-1,$$

where $v_i = (q^i - 1)/(q - 1)$.

- probably hopeless in all generality
- Belov, Logachev, Sandimirov, 1974
- N. Hamada,T. Helleseth
- R. Hill
- T. Maruta
- L. Storme, J. De Beule, P. Govaerts et al.
- A. Klein, Kl. Metsch and many others

3. Characterization of the $(102, 26)$-arcs in $\mathrm{PG}(3, 4)$

3.1. The $(101, 26)$- and $(102, 26)$-arcs in $\mathrm{PG}(3, 4)$

Theorem. There exists exactly one $(102, 26)$-arc in $\mathrm{PG}(3, 4)$. It is obtained as the sum of an ovoid and the complete space.

Spectrum:

$$a_{22} = 17, a_{26} = 68, \quad \lambda_0 = 0, \lambda_1 = 68, \lambda_2 = 17.$$

Theorem. Every $(101, 26)$-arc in $\mathrm{PG}(3, 4)$ is extendable to a $(102, 26)$-arc.

d	$g_4(5,d)$	$n_4(5,d)$	(n,w)-arc $\mathcal{K}$	$\mathcal{K}\|_H$
297	398	399	$(399,101)$-arc	$(101,26)$-arc in $\mathrm{PG}(3,4)$
298	399	400	$(400,101)$-arc	
299	400	401	$(401,101)$-arc	
300	401	402	$(402,101)$-arc	
301	403	404	$(404,102)$-arc	$(102,26)$-arc in $\mathrm{PG}(3,4)$
302	404	405	$(405,102)$-arc	
303	405	406	$(406,102)$-arc	
304	406	407	$(407,102)$-arc	

Open problem. Characterize geometrically the arcs with parameters

$$(q^3+2q^2+q+2, q^2+2q+2) \text{ in } \mathrm{PG}(3,q), q>2.$$

These arcs are associated with Griesmer codes with parameters

$$[q^3+2q^2+q+2, 4, q^3+q^2-q]_q.$$

An obvious construction: the sum of an ovoid and the whole space $\mathrm{PG}(3,q)$.

The question is: are there other constructions?

• **In $\mathrm{PG}(3,3)$:** We have two $(50,17)$-arcs:

(a) the sum of a cap and the whole space;

(b) two copies of $\mathrm{PG}(3,3)$ minus two different planes π_0, π_1 minus a line (skew to the line $\ell = \pi_0 \cap \pi_1$).

• **In $\mathrm{PG}(3,4)$:** There is exacly one $(102,26)$-arc and it is the sum of an ovoid and the whole space.

• **In $\mathrm{PG}(3,5)$:** There is exacly one $(182,37)$-arc and it is the sum of an ovoid and the whole space.

Conjecture. (At least) for every prime $p \geq 5$ there is a unique arc with parameters $(p^3 + 2p^2 + p + 2, p^2 + 2p + 2)$ in $\mathrm{PG}(3, p)$. It is obtained as the sum of an ovoid and the whole space.

How can one prove this?

3.2. Reducibility of plane $(x(q+1)+1, x)$-minihypers

The planes of maximal multiplicity have parameters $(q^2+2q+2, q+3)$.

The existence of such arcs is equivalent to that of minihypers with parameters $(q^2, q-1)$ (with maximal multiplicity of a point equal to 2).

These parameters can be written as $(x(q+1)+1, x)$ with $x = q-1$.

Reducible $(x(q+1)+1, x)$-minihypers. can be obtained from $(x(q+1), x)$-minihypers by adding a point.

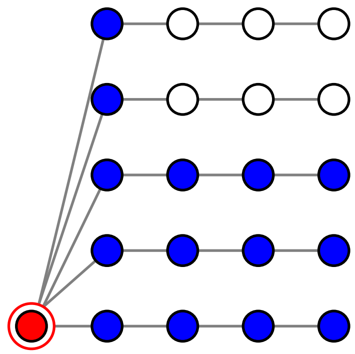
R. Hill, H.N Ward

I. Landjev, L. Storme

Irreducible $(x(q+1)+1, x)$-minihypers.

- the complement of an oval for all odd q
- for $q = 4$: one irreducible $(16, 3)$-minihyper
- for $q = 5$: one further irreducible minihyper with $\lambda_2 = 2, \lambda_0 = 8$.

$(16, 3)$-minihyper in $\mathrm{PG}(2, 4)$



$(25, 4)$-minihyper in $\mathrm{PG}(2, 5)$