New ternary linear codes of dimension 6

Tatsuya Maruta

(Joint work with Yusuke Oya)

Department of Mathematics
and Information Sciences

Osaka Prefecture University

maruta@mi.s.osakafu-u.ac.jp

Overview

We show how to construct new ternary linear codes with parameters

$[385, 6, 255]_3$, $[389, 6, 258]_3$, $[393, 6, 261]_3$, $[398, 6, 264]_3$, $[402, 6, 267]_3$, $[457, 6, 303]_3$, $[466, 6, 309]_3$, $[470, 6, 312]_3$

from a $[406, 6, 270]_3$ code which was found by Takenaka-Okamoto-M (2008).

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1. Optimal linear codes problem

$\mathbb{F}_q^n = \{(a_1, a_2, ..., a_n) \mid a_1, ..., a_n \in \mathbb{F}_q\}$.
For $a = (a_1, ..., a_n), b = (b_1, ..., b_n) \in \mathbb{F}_q^n$,
the (Hamming) distance between a and b is

$$d(a, b) = |\{i \mid a_i \neq b_i\}|.$$

The weight of a is $wt(a) = |\{i \mid a_i \neq 0\}| = d(a, \mathbf{0})$.

An $[n, k, d]_q$ code $\mathcal{C}$ means a k-dimensional subspace of $\mathbb{F}_q^n$ with minimum distance d,

$$\begin{aligned} d &= \min\{d(a, b) \mid a \neq b,\ a, b \in \mathcal{C}\} \\ &= \min\{wt(a) \mid wt(a) \neq 0,\ a \in \mathcal{C}\}. \end{aligned}$$

The elements of $\mathcal{C}$ are called codewords.

A good $[n, k, d]_q$ code will have

small n for fast transmission of messages,

large k to enable transmission of a wide variety of messages,

large d to correct many errors.

Optimal linear codes problem.

Optimize one of the parameters n, k, d for given the other two.

Optimal linear codes problem.

Problem 1. Find $n_q(k,d)$, the smallest value of n for which an $[n,k,d]_q$ code exists.

Problem 2. Find $d_q(n,k)$, the largest value of d for which an $[n,k,d]_q$ code exists.

An $[n,k,d]_q$ code is called optimal if

$$n = n_q(k,d) \text{ or } d = d_q(n,k).$$

As for the updated bounds on $d_q(n,k)$ for small q, k, n see the website maintained by Markus Grassl:

http://www.codetables.de/.

Optimal linear codes problem.

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An $[n,k,d]_q$ code is called optimal if

$$n = n_q(k,d) \text{ or } d = d_q(n,k).$$

See also

http://www.geocities.jp/mars39geo/griesmer.htm
for $n_q(k,d)$ tables for some small q and k.

The Griesmer bound

$$n_q(k,d) \geq g_q(k,d) := \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil$$

where $\lceil x \rceil$ is a smallest integer $\geq x$.

Griesmer (1960) proved for binary codes.
Solomon and Stiffler (1965) proved for all q.

A linear code attaining the Griesmer bound is called a Griesmer code.
Griesmer codes are optimal.

Problem to determine $n_3(k, d)$ for all d

$[k \leq 5]$

$n_3(k, d) = g_3(k, d)$ for all d for $k = 1, 2$.

$n_3(3, d) = g_3(3, d)$ for all $d \neq 3$,

$n_3(3, 3) = g_3(3, 3) + 1$.

$n_3(4, d) = g_3(4, d) + 1$ for $d = 3$, 7-9, 13-15,

$n_3(4, d) = g_3(4, d)$ for other d.

$n_3(5, d) = g_3(5, d) + 1$ for

$d = 3$, 7-9, 13-24, 32, 33, 37-51, 61-63, 94-99,

$n_3(5, d) = g_3(5, d) + 2$ for $d = 25$-27,

$n_3(5, d) = g_3(5, d)$ for other d.

Problem to determine $n_3(k, d)$ for all d

[$k \leq 5$]
Hill-Newton (1992) solved for
$k \leq 4$ for all d and $k = 5$ for all but 30 values of d.
van Eupen, Bogdanova, Boukliev, Hamada, Helleseth, etc. solved partially for $k = 5$ and Landjev (1998) completed for the remaining values of d.

[$k = 6$]
Hamada (1993) tackled for $d \leq 243$.
Takenaka-Okamoto-M (2008) tackled for $d > 243$.
$n_3(6, d)$ is still undetermined for 136 values of d.

Put $g = g_3(6, d)$. It is known that

$n_3(6, d) = g$ or $g + 1$ for $d = 175, 200$,253-267,

$n_3(6, d) = g + 1$ or $g + 2$ for $d =$ 310-312,

$g \le n_3(6, d) \le g + 2$ for $d = 302, 303,$ 307-309.

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We prove

$n_3(6, d) = g$ for $d =$ 253-267,

$n_3(6, d) = g + 1$ for $d =$ 175, 200, 310-312,

$n_3(6, d) = g$ or $g + 1$ for $d =$ 302, 303, 307-309

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$n_3(6, d) = g$ or $g + 1$ for $d = 175, 200$, 253-267,

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We prove

$n_3(6, d) = g$ for $d =$ 253-267,

$n_3(6, d) = g + 1$ for $d = 175, 200,$ 310-312,

$n_3(6, d) = g$ or $g + 1$ for $d = 302, 303,$ 307-309

by showing that

$\exists [g_3(6, d), 6, d]_3$ for $d =$ 253-267,

$\exists [g_3(6, d) + 1, 6, d]_3$ for $d = 302, 303,$ 307-312,

$\nexists [g_3(6, d), 6, d]_3$ for $d = 175, 200$.

Since $\exists[n,k,d]_q \Rightarrow \exists[n-1,k,d-1]_q$
we construct
$[g_3(6,d),6,d]_3$ for $d=255,258,261,264,267$
and
$[g_3(6,d)+1,6,d]_3$ for $d=303,309,312$.

The nonexistence of $[g_3(6,d),6,d]_3$ for $d=175,200$
will be shown in the next talk by Oya.

Note. We have recently proved
$\nexists[g_3(6,d),6,d]_3$ for $d=302,303,308,309$.
This implies that
$n_3(6,d)=g_3(6,d)+1$ for $d=302,303,308,309$.
Now $n_3(6,d)$ is still undetermined for 112 values of d.

2. A geometric approach

$\mathrm{PG}(r, q)$: projective space of dim. r over $\mathbb{F}_q$

j-flat: j-dim. projective subspace of $\mathrm{PG}(r, q)$

$\theta_j := |\mathrm{PG}(j, q)| = (q^{j+1} - 1)/(q - 1)$

$\mathcal{C}$: an $[n, k, d]_q$ code with $B_1 = 0$
i.e. with no coordinate which is identically zero

G: a generator matrix of $\mathcal{C}$
The columns of G can be considered as a multiset of n points in $\Sigma = \mathrm{PG}(k - 1, q)$ denoted also by $\mathcal{C}$.

$\mathcal{F}_j :=$ the set of j-flats of Σ

$\Sigma \ni P$: i-point $\Leftrightarrow$ P has multiplicity i in $\mathcal{C}$

$\gamma_0 = \max\{i \mid \exists P : i\text{-point in } \Sigma\}$

$C_i := \{P \in \Sigma \mid P : i\text{-point}\}$, $0 \le i \le \gamma_0$

For ${}^\forall S \subset \Sigma$ we define the multiplicity of S, denoted by $m_{\mathcal{C}}(S)$, as

$$m_{\mathcal{C}}(S) = \sum_{i=1}^{\gamma_0} i \cdot |S \cap C_i|.$$

Then we obtain the partition $\Sigma = \bigcup_{i=0}^{\gamma_0} C_i$ s.t.

$$\begin{aligned} n &= m_{\mathcal{C}}(\Sigma), \\ n - d &= \max\{m_{\mathcal{C}}(\pi) \mid \pi \in \mathcal{F}_{k-2}\}. \end{aligned}$$

Conversely such a partition of Σ as above gives an $[n, k, d]_q$ code in the natural manner.

For a t-flat Π in Σ we define

$$\gamma_j(\Pi) = \max\{m_{\mathcal{C}}(\Delta) \mid \Delta \subset \Pi,\ \Delta \in \mathcal{F}_j\},\ 0 \le j \le t.$$

We denote simply by γ_j instead of $\gamma_j(\Sigma)$.

A line l is called an i-line if $m_{\mathcal{C}}(l) = i$.

An i-plane, an i-solid and so on are defined similarly.

$a_i = |\{H \in \mathcal{F}_{k-2} \mid m_{\mathcal{C}}(H) = i\}| =$ # of i-hps

List of a_i's: the spectrum of $\mathcal{C}$

Lemm 1

(1) $\sum_i a_i = \theta_{k-1}$. (2) $\sum_i i a_i = n\theta_{k-2}$.

(3) $\sum_i i(i-1)a_i = n(n-1)\theta_{k-3} + q^{k-2} \sum_{s \ge 2} s(s-1)\lambda_s$.

For a t-flat Π in Σ we define

$$\gamma_j(\Pi) = \max\{m_{\mathcal{C}}(\Delta) \mid \Delta \subset \Pi,\ \Delta \in \mathcal{F}_j\},\ 0 \le j \le t.$$

We denote simply by γ_j instead of $\gamma_j(\Sigma)$.
A line l is called an i-line if $m_{\mathcal{C}}(l) = i$.
An i-plane, an i-solid and so on are defined similarly.
Recall that $\gamma_{k-1} = n,\ \ \gamma_{k-2} = n - d$.

γ_j's are determined when $\mathcal{C}$ is Griesmer:

$$\gamma_j = \sum_{u=0}^{j} \left\lceil \frac{d}{q^{k-1-u}} \right\rceil \text{ for } 0 \le j \le k-1.$$

For a t-flat Π in Σ we define

$$\gamma_j(\Pi) = \max\{m_{\mathcal{C}}(\Delta) \mid \Delta \subset \Pi,\ \Delta \in \mathcal{F}_j\},\ 0 \le j \le t.$$

We denote simply by γ_j instead of $\gamma_j(\Sigma)$.
A line l is called an i-line if $m_{\mathcal{C}}(l) = i$.
An i-plane, an i-solid and so on are defined similarly.
Recall that $\gamma_{k-1} = n,\ \ \gamma_{k-2} = n - d$.

Lemma 2.

$$\gamma_j \le \gamma_{j+1} - \frac{n - \gamma_{j+1}}{\theta_{k-2-j} - 1} \quad \text{for } 0 \le j \le k-3.$$

3. Constructing new codes

Lemma 3.
$\mathcal{C}$: $[n,k,d]_q$ code, $\Sigma = \mathrm{PG}(k-1,q)$, $0 \le t \le k-2$
$\cup_{i=0}^{\gamma_0} C_i$: the partition of Σ obtained from $\mathcal{C}$.
$\cup_{i \ge 1} C_i \supset \Delta$: t-flat, $\nexists H$: hp s.t. $H \supset (\cup_{i\ge 1} C_i) \setminus \Delta$
$\Rightarrow \quad \exists \mathcal{C}'$: $[n-\theta_t, k, d-q^t]_q$ code

Proof. Define a new partition $\Sigma = \cup_i C_i'$ by

$$C_i' = (C_i \setminus \Delta) \cup (C_{i+1} \cap \Delta) \text{ for all } i$$

which gives an $[n' = n - \theta_t, k, d']_q$ code $\mathcal{C}'$.
For $\forall H \in \mathcal{F}_{k-2}$, $H \cap \Delta = \theta_{t-1}$ or θ_t.
So, $m_{\mathcal{C}'}(H) \le n' - d' \le n - d - \theta_{t-1}$, giving $d' \ge d - q^t$.

3. Constructing new codes

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Example.

$\mathcal{C}$: simplex $[\theta_{k-1}, k, q^{k-1}]_q$ code

Δ: a hp of Σ

$\Rightarrow$ $\mathcal{C}'$: Griesmer $[q^{k-1}, k, q^{k-1}-q^{k-2}]_q$ code

3. Constructing new codes

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$\cup_{i \geq 1} C_i \supset \Delta$: t-flat, $\nexists H$: hp s.t. $H \supset (\cup_{i \geq 1} C_i) \setminus \Delta$
$\Rightarrow \quad \exists \mathcal{C}'$: $[n - \theta_t, k, d - q^t]_q$ code

Note.

The converse of Lemma 3 holds if $\exists \Delta$: t-flat s.t.

$$m_{\mathcal{C}}(H) \leq n - d - \theta_t \text{ for all hp } H \supset \Delta.$$

K: an n-set in PG(r, q), $r \geq 3$

K is an n-cap $\Leftrightarrow$ $|K \cap l| \leq 2$ for all line l.

$m_2(r, q)$= max$\{n \mid \exists K$: n-cap in PG$(r, q)\}$

The following results are known for $q = 3$:

Lemma 4.

(1) $m_2(3, 3) = 10$ (Bose, 1947)

(2) $m_2(4, 3) = 20$ (Pellgrino, 1970)

(3) $m_2(5, 3) = 56$ (Hill, 1973)

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A set $\mathcal{B}$ in PG$(2, q)$ is a blocking set if

$$l \cap \mathcal{B} \neq \emptyset \text{ for any line } l.$$

$\mathcal{B}$ is non-trivial if it contains no line.

$b(q) := \min\{b \mid \exists\mathcal{B}\text{: non-trivial blocking set in PG}(2, q)\}$

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A set $\mathcal{B}$ in PG(r,q) is a blocking set w.r.t. s-flats if

$$S \cap \mathcal{B} \neq \emptyset \text{ for any } s\text{-flat } S \text{ in PG}(r,q).$$

A blocking set in PG(r,q) with respect to s-flats is non-trivial if it contains no $(r-s)$-flat.

Theorem 5 (Bose-Burton(1966), Beutelspacher(1980))
Let $\mathcal{B}$ be a blocking set w.r.t. s-flats in PG(r,q).

(1) $|\mathcal{B}| \geq \theta_{r-s}$ and

$$|\mathcal{B}| = \theta_{r-s} \iff \mathcal{B} \text{ is an } (r-s)\text{-flat.}$$

(2) $|\mathcal{B}| \geq \theta_{r-s} + q^{r-s-1}(b(q) - \theta_1)$ if $\mathcal{B}$ is non-trivial.

We construct codes with parameters

$$[385, 6, 255]_3,\ [389, 6, 258]_3,\ [393, 6, 261]_3,$$
$$[398, 6, 264]_3,\ [402, 6, 267]_3$$

from a $[406, 6, 270]_3$ code with spectrum

$$(a_{82}, a_{109}, a_{136}) = (1, 12, 351)$$

found by Takenaka et al. (2008).

a 109-hp $\longleftrightarrow$ a $[109, 5, 72]_3$ code

a 136-hp $\longleftrightarrow$ a $[136, 5, 90]_3$ code

We first investigate $[109, 5, 72]_3$ and $[136, 5, 90]_3$ codes.

Lemm 6.

$\mathcal{C}$: $[109, 5, 72]_3$ code

$\Rightarrow \quad \gamma_0 \leq 2$ and $\gamma_1 \leq 5$ by Lemma 2.

Assume $a_i = 0$ for all $i \notin \{1, 10, 19, 28, 37\}$.

Then the partition of $\Sigma = \mathrm{PG}(4, 3)$ from $\mathcal{C}$ satisfies

(1) $C_1 \cup C_2$ contains two skew lines.

(2) For any line $l_1 \subset C_1 \cup C_2$,

$\exists l_2, l_3 \subset C_1 \cup C_2$ s.t. l_1, l_2, l_3 are skew.

Lemm 6.

$\mathcal{C}$: $[109, 5, 72]_3$ code

$\Rightarrow$ $\gamma_0 \le 2$ and $\gamma_1 \le 5$ by Lemma 2.

Assume $a_i = 0$ for all $i \notin \{1, 10, 19, 28, 37\}$.

Proof.

Let $\lambda_i = |C_i|$ for $0 \le i \le 2$.

From $\lambda_0 + \lambda_1 + \lambda_2 = \theta_4$, $\lambda_1 + 2\lambda_2 = n$, we get

$\lambda_2 = \lambda_0 - 12$.

C_2 forms a λ_2-cap, for $\gamma_1 \le 5$.

Hence $\lambda_2 \le 20$ (from $m_2(4,3) = 20$) and $\lambda_0 \le 32$.

Lemma 1.

(1) $\sum_i a_i = \theta_{k-1}$. (2) $\sum_i i a_i = n\theta_{k-2}$.

(3) $\sum_i i(i-1)a_i = n(n-1)\theta_{k-3} + q^{k-2} \sum_{s \geq 2} s(s-1)\lambda_s$.

From Lemma 1, we get

$$6a_1 + 3a_{10} + a_{19} = \lambda_2/3 - 4$$

which implies $3|\lambda_2$. Hence $3|\lambda_0 (= \lambda_2 + 12)$.
This improves $\lambda_0 \leq 32$ to

$$|C_0| = \lambda_0 \leq 30.$$

Theorem 5.

Let $\mathcal{B}$ be a blocking set w.r.t. s-flats in PG(r, q).

(1) $|\mathcal{B}| \geq \theta_{r-s}$

$\mathcal{B}$: blocking set w.r.t. lines in PG$(4, 3)$ $\Rightarrow$ $\mathcal{B} \geq 40$

Hence C_0 is not a blocking set w.r.t. lines

$\Rightarrow \quad \exists l_1 \subset C_1 \cup C_2$.

Since $|C_0 \cup l_1| \leq 30 + 4 < 40$,

$\exists l_2$: a line which is disjoint from $C_0 \cup l_1$.

Since $|C_0 \cup l_1 \cup l_2| \leq 34 + 4 < 40$,

$\exists l_3$: a line which is disjoint from $C_0 \cup l_1 \cup l_2$. $\square$

Theorem 7 (Ward, 1998).

$\mathcal{C}$: a Griesmer $[n,k,d]_p$ code, p a prime.

$p^e|d \;\Rightarrow\; p^e|w$ for all $A_w > 0$.

Lemma 8.

$\mathcal{C}$: Griesmer $[136,5,90]_3$ code

$C_0 \cup C_1 \cup C_2$: the partition of $\Sigma = \mathrm{PG}(4,3)$ from $\mathcal{C}$.

Then

(1) $a_i = 0$ for all $i \notin \{10,19,28,37,46\}$.

(2) $C_1 \cup C_2$ contains a plane if $\lambda_0 = |C_0| \leq 18$.

Lemma 8.

$\mathcal{C}$: Griesmer $[136, 5, 90]_3$ code

$C_0 \cup C_1 \cup C_2$: the partition of $\Sigma = \mathrm{PG}(4, 3)$ from $\mathcal{C}$.

(1) $a_i = 0$ for all $i \notin \{10, 19, 28, 37, 46\}$.

Proof.

(1) $a_i = 0$ for all $i \notin \{1, 10, 19, 28, 37, 46\}$,

since $9|w$ for all $A_w > 0$ by Theorem 7 (Ward).

Considering the solids through the 1-plane in a putative 1-solid, one can get a contradiction.

Hence $a_1 = 0$.

Thm 5

(2) $|\mathcal{B}| \geq \theta_{r-s} + q^{r-s-1}(b(q) - \theta_1)$ if $\mathcal{B}$ is non-trivial.

Lemma 8.

$\mathcal{C}$: Griesmer $[136, 5, 90]_3$ code

$C_0 \cup C_1 \cup C_2$: the partition of $\Sigma = \mathrm{PG}(4, 3)$ from $\mathcal{C}$.

(1) $a_i = 0$ for all $i \notin \{10, 19, 28, 37, 46\}$.

(2) $C_1 \cup C_2$ contains a plane if $\lambda_0 = |C_0| \leq 18$.

Proof.

(2) It can be checked: C_0 contains no plane.

Suppose $C_1 \cup C_2$ contains no plane.

$\Rightarrow$ C_0 forms a non-trivial blocking set w.r.t. planes.

$\Rightarrow$ $|C_0| \geq \theta_2 + 3(6 - 4) = 19$, a contradiction. □

$\mathcal{C}$: Griesmer $[406, 6, 270]_3$ code with spectrum

$$(a_{82}, a_{109}, a_{136}) = (1, 12, 351)$$

$C_0 \cup C_1 \cup C_2$: the partition of $\Sigma = \mathrm{PG}(5, 3)$ from $\mathcal{C}$

$\Rightarrow \quad (\lambda_0, \lambda_1, \lambda_2) = (51, 220, 93)$, where $\lambda_i = |C_i|$.

Π_0: a 136-hp $\longleftrightarrow$ a $[136, 5, 90]_3$ code.

Π_1: a 109-hp $\longleftrightarrow$ a $[109, 5, 72]_3$ code

$j \in \{10, 19, 28, 37, 46\}$ for any j-solid in Π_0

$\Rightarrow j \in \{1, 10, 19, 28, 37\}$ for any j-solid in Π_1

$\Rightarrow \exists l_1, l_2$: skew lines in $\Pi_1 \cap (C_1 \cup C_2)$ (by Lemma 6(1))

$\Rightarrow \exists [402, 6, 267]_3$ and $[398, 6, 264]_3$ (by Lemma 3)

We constructed $\mathcal{C}$ as a projective dual of a $[14, 6, 6]_3$ code $\mathcal{C}^*$ with a generator matrix

$$G^* = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 2 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 2 & 1 & 1 & 1 \end{bmatrix}.$$

$\mathcal{C}^*$ has spectrum

$$(a_2, a_5, a_8) = (\lambda_2, \lambda_1, \lambda_0) = (93, 220, 51).$$

$C_0 = \{x_1 \cdots x_6 \in \Sigma \mid wt(x_1 g_1 + \cdots + x_6 g_6) = 6\}$
where g_i is the i-th row of G^*.
C_0 in $\Sigma = \mathrm{PG}(5, 3)$ is obtained from G^* as follows.

$C_0 = \{100000, 010000, 001000, 011000, 121000, 012000, 000100, 010100, 001100, 011100, 100200, 010200, 001200, 011200, 012200, 000010, 100010, 010010, 001010, 101010, 012010, 000110, 001110, 120210, 010020, 010120, 001120, 000001, 010001, 001001, 010201, 000011, 010011, 001011, 011011, 000111, 001021, 100002, 120002, 001002, 012002, 000102, 010202, 001202, 101202, 010012, 000112, 011022, 000122, 010122, 001222\}$,

$b_i :=$ # of hps Π of Σ with $|\Pi \cap C_0| = i$

Then, we get

$$(b_{42}, b_{27}, b_{24}, b_{21}, b_{18}, b_{15}) = (1, 12, 12, 12, 120, 207). \qquad (1)$$

Recall $(a_{82}, a_{109}, a_{136}) = (1, 12, 351)$.

Π_2: 82-hp (contains at least 39 0-points)

Π_2 contains exactly 42 0-points from (1).

We checked: the 109-hps contain exactly 27 0-points.

$\therefore H$: hp, $H \cap C_0 = 18$ or 15

$\Rightarrow H$: 136-hp

$\Rightarrow H$ has a plane contained in $C_1 \cup C_2$ by Lemma 8

Since # of 4-flats through a fixed plane in Σ is θ_2,

$\exists \Pi_1$: 136-hp through a plane $\delta_1 \subset C_1 \cup C_2$ and

$\exists \Pi_2$: a 109-hp s.t. $\Pi_2 \cap \delta = l_1$: a line.

Actually, taking

$$\delta = \langle 120000, 001210, 110111 \rangle \subset C_1 \cup C_2,$$

all of the 4-flats $\supset \delta$ are 136-hps, and removing δ (Lemma 3) gives a $[393, 6, 261]_3$ code with

$$(a_{78}, a_{105}, a_{123}, a_{132}) = (1, 12, 13, 338).$$

Since $\Pi_2 \cap \delta = l_1$, we can take two lines l_2 and l_3 in Π_2 s.t. l_1, l_2, l_3 are skew, $l_1 \cup l_2 \cup l_3 \subset C_1 \cup C_2$ by Lemma 6(2).
Hence we get $[389, 6, 258]_3$ and $[385, 6, 255]_3$ codes applying Lemma 3 again.

Taking $l_2 = \langle 010101, 100001 \rangle$,
we get a $[389, 6, 258]_3$ code with spectrum

$$(a_{77}, a_{101}, a_{104}, a_{119}, a_{122}, a_{128}, a_{131}) = (1, 2, 10, 1, 12, 37, 301),$$

and taking $l_3 = \langle 110000, 000101 \rangle$
gives a $[385, 6, 255]_3$ code with spectrum

$$(a_{76}, a_{97}, a_{103}, a_{118}, a_{121}, a_{124}, a_{127}, a_{130}) = (1, 2, 10, 2, 11, 2, 70, 266).$$

$[457, 6, 303]_3$ and $[470, 6, 312]_3$ codes are obtained from these codes applying the following lemma:

Lemma 9.

$\mathcal{C}_1$: $[n_1, k, d_1]_q$, $\mathcal{C}_2$: $[n_2, k-1, d_2]_q$

$\exists c \in \mathcal{C}_1$ with $wt(c) \geq d_1 + d_2$

$\Rightarrow$ $\exists \mathcal{C}_3$: $[n_1 + n_2, k, d_1 + d_2]_q$

$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$
$[402, 6, 267]_3$	$[55, 5, 36]_3$	$[457, 6, 303]_3$
$[385, 6, 255]_3$	$[81, 5, 54]_3$	$[466, 6, 309]_3$
$[389, 6, 258]_3$	$[81, 5, 54]_3$	$[470, 6, 312]_3$

Note. The $[385, 6, 255]_3$ and $[389, 6, 258]_3$ codes have codewords with weight 309 and 312, respectively.

Since $\exists[n,k,d]_q \Rightarrow \exists[n-1,k,d-1]_q$
we construct

$[g_3(6,d),6,d]_3$ for $d = 255, 258, 261, 264, 267$

and

$[g_3(6,d)+1,6,d]_3$ for $d = 303, 309, 312$.

The nonexistence of $[g_3(6,d),6,d]_3$ for $d = 175, 200$
will be shown in the next talk by Oya.

Note. We have recently proved
$\nexists[g_3(6,d),6,d]_3$ for $d = 302, 303, 308, 309$.
This implies that
$n_3(6,d) = g_3(6,d)+1$ for $d = 302, 303, 308, 309$.
Now $n_3(6,d)$ is still undetermined for 112 values of d.

Thank you for your attention!

Lemma 10 (Takenaka-Okamoto-M, 2008).
A Griesmer $[406, 6, 270]_3$ code exists and its spectrum is one of the following:

(a) $(a_{82}, a_{109}, a_{136}) = (1, 12, 351)$,
with $(\lambda_2, \lambda_1, \lambda_0) = (93, 220, 51)$,

(b) $(a_{82}, a_{109}, a_{136}) = (2, 10, 352)$,
with $(\lambda_2, \lambda_1, \lambda_0) = (102, 202, 60)$,

(c) $(a_{109}, a_{136}) = (14, 350)$,
with $(\lambda_2, \lambda_1, \lambda_0) = (84, 238, 42)$.