The nonexistence of $[265, 6, 175]_3$ codes and $[302, 6, 200]_3$ codes

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(Joint work with T. Maruta)

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0. Introduction

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$$n_3(6,d) := \min\{n \mid \exists [n,6,d]_3 \text{ code}\}$$

$$g_3(6,d) := \sum_{i=0}^{5} \lceil d/3^i \rceil$$

0. Introduction

d	$g_3(6,d)$	$n_3(6,d)$
175	265	265-266
200	302	302-303

Problem. Do $[265, 6, 175]_3$ codes and $[302, 6, 200]_3$ codes exist ?

0. Introduction

d	$g_3(6,d)$	$n_3(6,d)$
175	265	265-266
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Problem. Do $[265, 6, 175]_3$ codes and $[302, 6, 200]_3$ codes exist ?

If such codes exist, then they are projective, that is, any two columns of a generator matrix are linearly independent.

Contents

1. The nonexistence of $[265, 6, 175]_3$ codes

Let C be a $[265, 6, 175]_3$ code.

The columns of a generator matrix of C can be considered as 265 points in Σ =PG(5, 3).

Let C_1 be the 265-set of Σ. $C_0 := \Sigma \setminus C_1$

An i-pt is a point of C_i.

There exists an $(n-d)$-hp in Σ.

$[265, 6, 175]_3$code $\rightarrow n - d = 90$

$\longrightarrow$ What is a code corresponding to a 90-hp?

1.1 The spectrum of a 90-hp

Lemma 1. Let Π be an i-hp through a t-secundum δ. $t = \max\{|\delta' \cap C_1| \mid \delta' \subset \Pi, \delta' \in \mathcal{F}_{k-3}\}$, then

$$t \leq \frac{i + q \cdot (n-d) - n}{q}$$

And an i-hp corresponds to a $[i, k-1, i-t]_3$ code.

Note. A secundum is a projective subspace of Σ of codimension 2.

Ex. A $[265, 6, 175]_3$ code

$$t \leq \frac{i + 3 \cdot 90 - 265}{3} = \frac{i + 5}{3}$$

$i = 90 \Rightarrow 90 - \left\lfloor \frac{90+5}{3} \right\rfloor = 59$

A 90-hp corresponds to a Griesmer $[90, 5, 59]_3$ code.

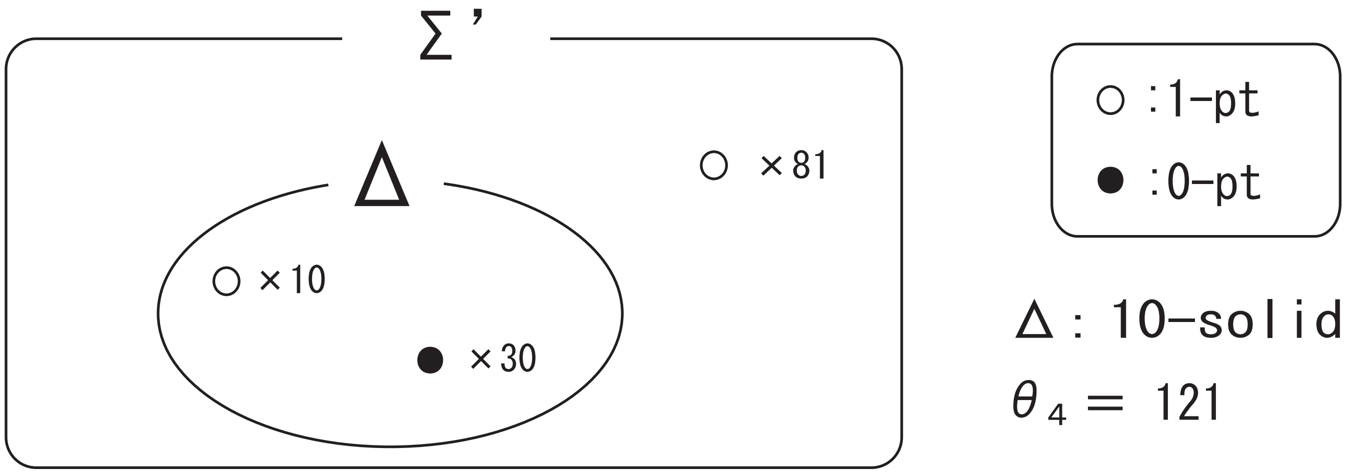
Extendability

For an $[n, k, d]_q$ code C with a generator matrix G, C is extendable if $[G, h]$ generates an $[n+1, k, d+1]_q$ code for some column vector h, $h^{\mathsf{T}} \in \mathbb{F}_q^k$.

Lemma 2 (Yoshida-Maruta, 2009).

Every $[90, 5, 59]_3$ code is extendable.

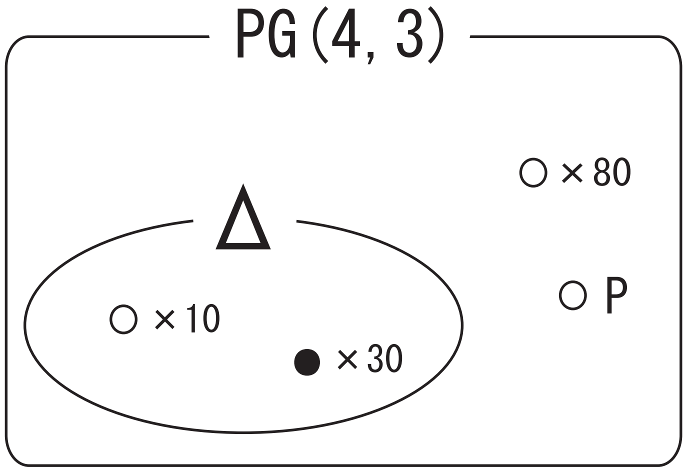
0-pts and 1-pts in $\Sigma' = PG(4,3)$ for a $[91,5,60]_3$ code



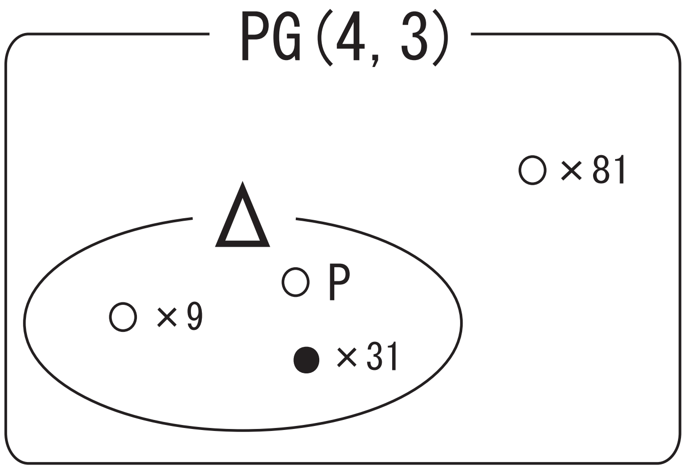
The set of 1-pts in Δ forms a 10-cap.

Since a $[90, 5, 59]_3$ code is extendable, it is obtained by changing a 1-pt P to the 0-pt in Σ'.

(1)

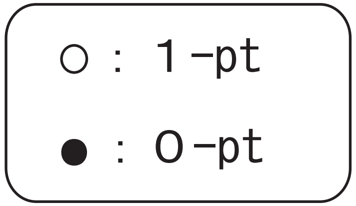


(2)

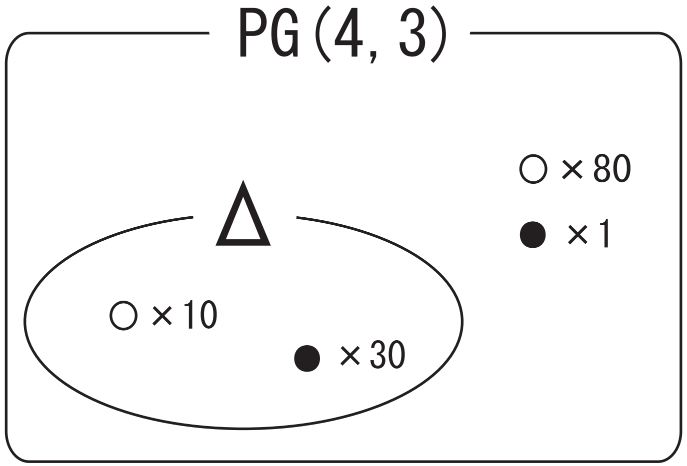


(1) $P \in PG(4,3) \setminus \Delta$.

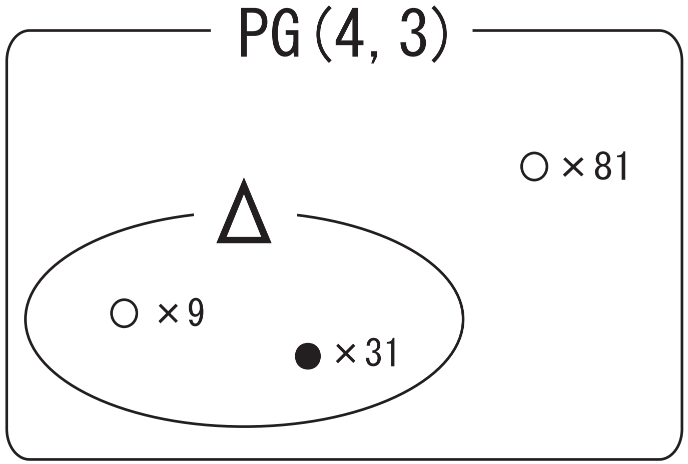
(2) $P \in \Delta$.



(1)



(2)



Lemma 3. The spectrum of a $[90, 5, 59]_3$ code is one of the following:

(1) $(\tau_{10}, \tau_{27}, \tau_{28}, \tau_{30}, \tau_{31}) = (1, 10, 20, 30, 60)$

(2) $(\tau_9, \tau_{27}, \tau_{28}, \tau_{30}, \tau_{31}) = (1, 3, 27, 36, 54)$

1.2 Proof

C : a $[265, 6, 175]_3$ code

The spectrum of a 90-hp in Σ =PG$(5, 3)$ satisfies

$$\tau_j > 0 \Rightarrow \ j \in \{9, 10, 27, 28, 30, 31\}$$

by Lemma 3.

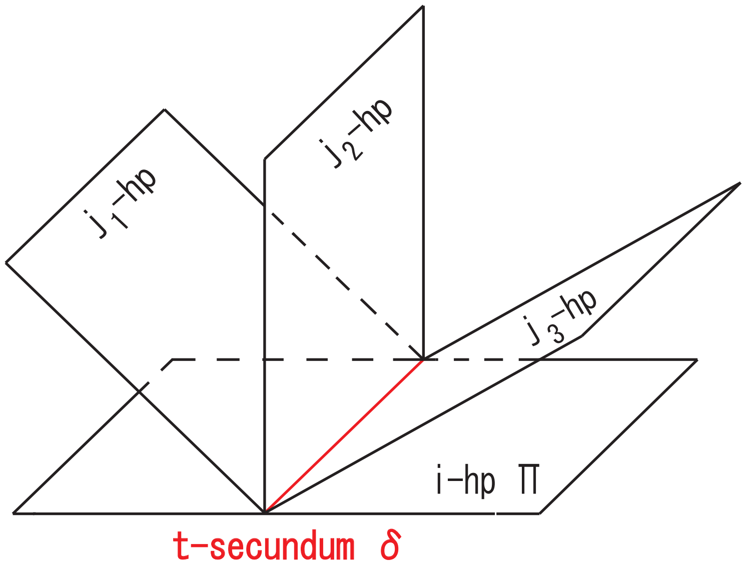
Hence the spectrum of C satisfies

$$a_i > 0 \ \Rightarrow \ i \in \{25, 55, 76\text{-}82, 85\text{-}90\}$$

by Lemma 1 and the known $n_3(5, d)$-table.

Lemma 4. Let Π be an i-hp through a t-secundum δ. Let c_j be the number of j-hps ($\neq \Pi$) through δ. Then

$$\sum_j ((n-d)-j)c_j = i + q\cdot(n-d) - n - qt$$



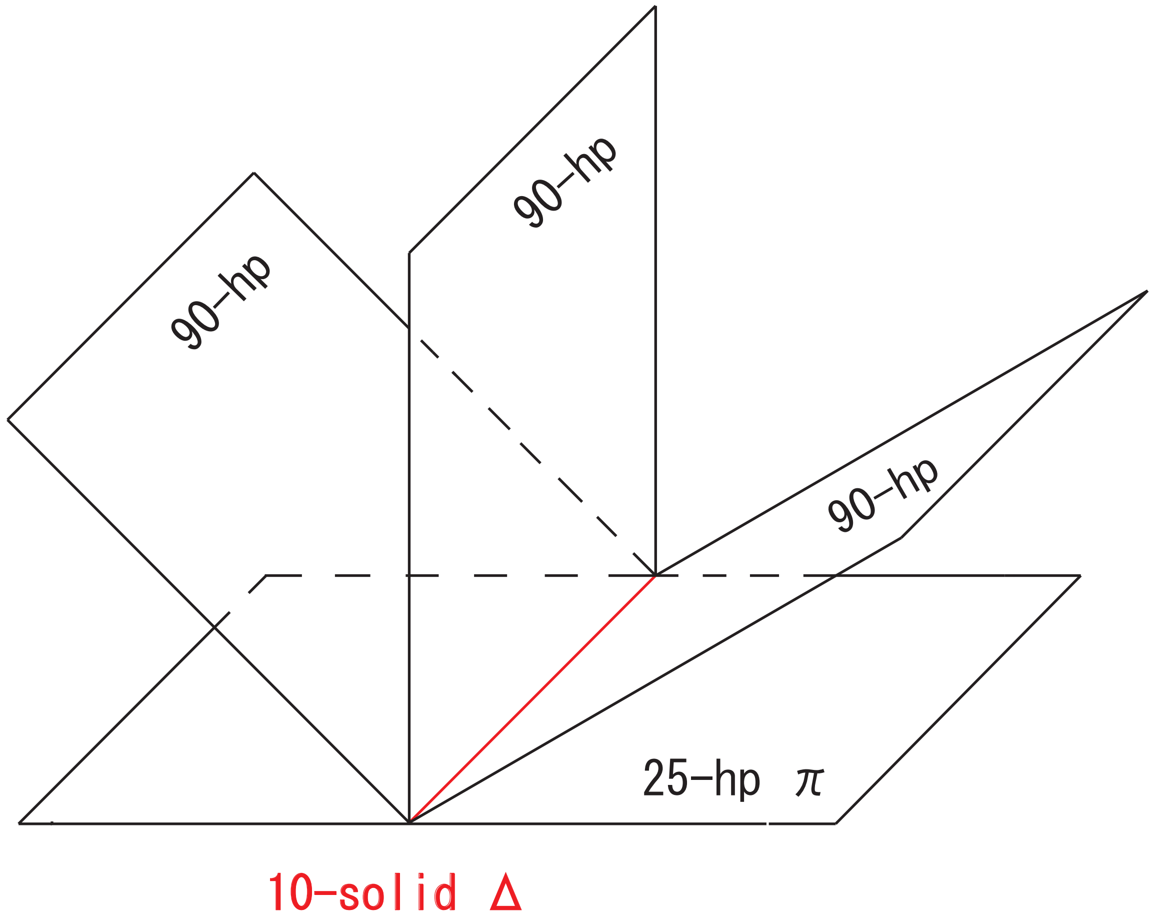
Suppose $a_{25} > 0$ and let π be a 25-hp.

Then π gives a $[25, 5, 15]_3$ code by Lemma 1.

$\Rightarrow$ π has a 10-solid. $(25 - 15 = 10)$

Let $\triangle$ be a 10-solid in π, then the hps through $\triangle$ are as the next slide by using Lemma 4.

The hps through a 10-solid Δ in 25-hp



PG(5, 3)

What is $\Delta \cap C_1$?

C_1 : the set of 1-pts in Σ =PG$(5, 3)$

(a) Δ' : 10-solid $\subset$ 90-hp

$\Rightarrow$ $\Delta' \cap C_1$: 10-cap

(by Lemma 1)

What is $\Delta \cap C_1$?

C_1 : the set of 1-pts in Σ =PG$(5,3)$

(a) Δ' : 10-solid $\subset$ 90-hp

$\Rightarrow$ $\Delta' \cap C_1$: 10-cap

(by Lemma 1)

(b) 10-solid $\Delta \subset$ 25-hp in PG$(5,3)$

$\Rightarrow \exists$ a 90-hp $\supset \Delta$

What is $\Delta \cap C_1$?

C_1 : the set of 1-pts in Σ =PG$(5, 3)$

(a) Δ' : 10-solid $\subset$ 90-hp

$\Rightarrow$ $\Delta' \cap C_1$: 10-cap

(by Lemma 1)

(b) 10-solid $\Delta \subset$ 25-hp in PG$(5, 3)$

$\Rightarrow \exists$ a 90-hp $\supset \Delta$

$\Rightarrow \Delta \cap C_1$: 10-cap, from (a)

$\Delta \cap C_1$: 10-cap $\Rightarrow$ Δ has 1-planes and 4-planes only.

Hence the spectrum of a 25-hp satisfies

$\tau_i > 0 \;\Rightarrow\; i \in \{7, 8, 9, 10\}$

by Lemma 1 and the known $n_3(4, d)$-table.

From Lemma 4 for $i = 10$, we obtain

$$3c_7 + 2c_8 + c_9 = 15 - 3t \qquad (1.1)$$

Then (1.1) has no solution for $t = 1$, a contradiction.

$\therefore\ a_{25} = 0$

Hence the spectrum of C satisfies

$a_i > 0 \quad \Rightarrow$

$$i \in \{25,55,76,77,78,79,80,81,82,85,86,87,88,89,90\}$$

Then, from Lemma 4 for $i = 90$, we obtain

$$35c_{55} + 14c_{76} + 13c_{77} + 12c_{78} + 11c_{79} + 10c_{80} + 9_{81}$$

$$+8c_{82} + 5c_{85} + 4c_{86} + 3c_{87} + 2c_{88} + c_{89} + 0c_{90} = 95 - 3t \qquad (1.2)$$

The spectrum of a 90-hp is one of the following:

(1) $(\underline{\tau_{10}}, \tau_{27}, \tau_{28}, \tau_{30}, \tau_{31}) = (1, 10, 20, 30, 60)$

(2) $(\underline{\tau_{9}}, \tau_{27}, \tau_{28}, \tau_{30}, \tau_{31}) = (1, 27, 28, 30, 54)$

by Lemma 3.

Then (1,2) has no solution for $\underline{t = 9, 10}$,

a contradiction.

$\therefore$ There exists no $[265, 6, 175]_3$ code.

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4. Results

2. Extendability and diversity

2.1 Definition of diversity

C : $[n, k, d]_q$ code, $\gcd(d, q) = 1$

The diversity (Φ_0, Φ_1) of C is given by

$$\Phi_0 = \frac{1}{q-1} \sum_{q \mid n-i} a_i \ , \quad \Phi_1 = \frac{1}{q-1} \sum_{i \not\equiv n, n-d \pmod{q}} a_i$$

2.2 Extendability for diversity

Theorem 1 (Hill-Lizak, 1999)

C: $[n, k, d]_q$ code, $\gcd(d, q) = 1$

$\Phi_1 = 0 \;\Rightarrow\; C$ is extendable.

Theorem 2 (Maruta, 2005)

C : $[n, 6, d]_3$ code, $\gcd(d, 3) = 1$

C is not extendable

$$\Rightarrow (\Phi_0, \Phi_1) \in \{(121, 81), (94, 135), (121, 108), (112, 126), (130, 117), (121, 135), (148, 108)\}$$

2.3 Definition of $(i, j)_t$ flat

C: projective $[n, k, d]_3$ code, $d \not\equiv 0 \pmod 3$, $k \geq 3$

Σ^*: the dual space of Σ =PG$(k-1, 3)$

$\mathcal{F}_j^*$: the set of j-flats in Σ^*

We define

$F_0 = \{ \Pi \in \mathcal{F}_0^* \mid |\Pi \cap C_1| \equiv n \pmod 3\}$,

$F_1 = \{ \Pi \in \mathcal{F}_0^* \mid |\Pi \cap C_1| \not\equiv n, n-d \pmod 3\}$,

$F_2 = \{ \Pi \in \mathcal{F}_0^* \mid |\Pi \cap C_1| \equiv n-d \pmod 3\}$.

C: $[n,k,d]_3$ code, (Φ_0, Φ_1): diversity of C

Then $a_i = |\{ \Pi \in \mathcal{F}_0^* \mid |\Pi \cap C_1| = i \}|$, hence

$$\Phi_0 = |F_0|, \quad \Phi_1 = |F_1|$$

Let S be a t-flat in Σ^*.

S is an $(i,j)_t$ flat if $|S \cap F_0| = i, |S \cap F_1| = j$.

Especially,

$(i,j)_1$ flats are (i,j)-lines

$(i,j)_2$ flats are (i,j)-planes.

Contents

3. The nonexistence of $[302, 6, 200]_3$ codes

There exists no $[303, 6, 201]_3$ code. (Hamada, 1995)

$$\Downarrow$$

A $[302, 6, 200]_3$ code is not extendable if it exists.

Let C be a $[302, 6, 200]_3$ code.

The columns of a generator matrix of C can be considered as 302 points in Σ =PG$(5, 3)$.

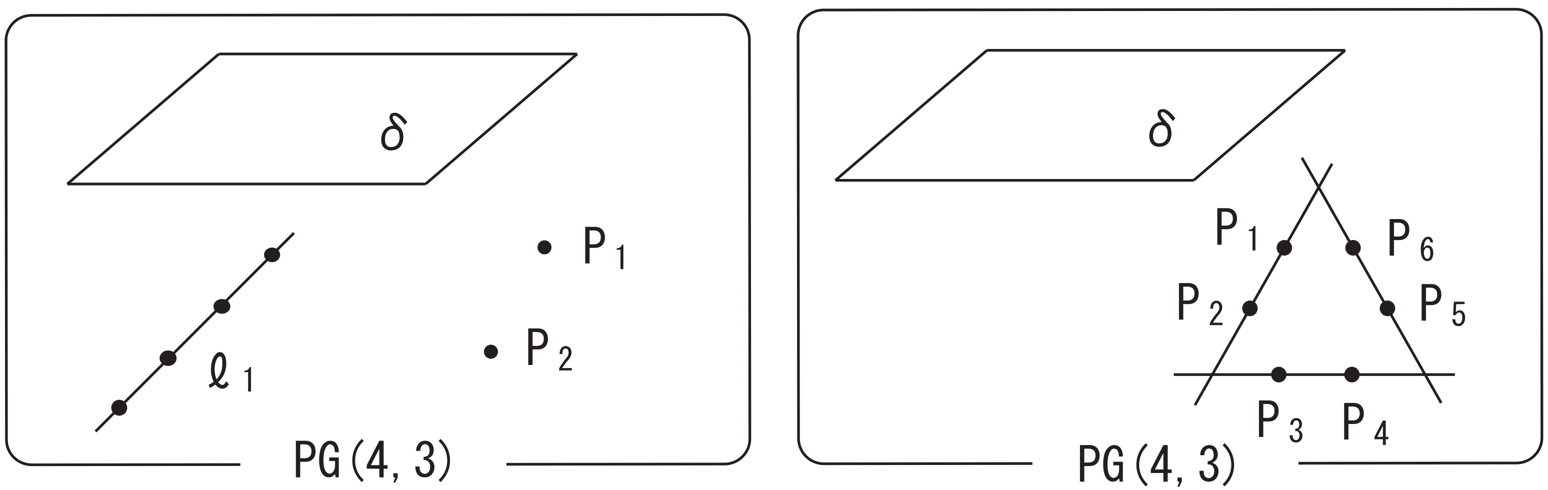
C_1: the 302-set $\subset \Sigma$, $C_0 := \Sigma \setminus C_1$

$\Rightarrow$ $^\exists\Pi$: a 102-hp in Σ. $(n - d = 102)$.

3.1 The spectrum of a 102-hp

Let Π be a 102-hp. Then $\Pi \cap C_0$ contains a plane δ and there are two possibilities for $(\Pi \cap C_0) \setminus \delta$:

(1) (2)



Lemma 5. The spectrum of a $[102, 5, 67]_3$ code is one of the following:

(1.1) $(a_{24}, a_{26}, a_{30}, a_{31}, a_{32}, a_{33}, a_{34}, a_{35}) = (1, 3, 1, 6, 6, 11, 48, 45)$

(1.2) $(a_{25}, a_{26}, a_{30}, a_{32}, a_{33}, a_{34}, a_{35}) = (2, 2, 4, 9, 9, 52, 43)$

(1.3) $(a_{25}, a_{26}, a_{30}, a_{31}, a_{32}, a_{33}, a_{34}, a_{35}) = (2, 2, 1, 6, 6, 12, 46, 46)$

(1.4) $(a_{24}, a_{26}, a_{30}, a_{33}, a_{34}, a_{35}) = (1, 3, 4, 35, 27, 51)$

(2.1) $(a_{24}, a_{26}, a_{30}, a_{32}, a_{33}, a_{34}, a_{35}) = (1, 3, 4, 9, 8, 54, 42)$

(2.2) $(a_{25}, a_{26}, a_{30}, a_{33}, a_{34}, a_{35}) = (2, 2, 4, 36, 25, 52)$

C : a $[302, 6, 200]_3$ code

The spectrum of C satisfies

$$a_i > 0 \;\Rightarrow\; i \in \{68, 69, 74, 80, 81, 86, 87, 89, 90,$$
$$91, 92, 95, 96, 98, 99, 100, 101, 102\}$$

Let (Φ_0, Φ_1) be the diversity of C.

C is not extendable $\Rightarrow$ $\Phi_1 = a_{91} + a_{100} > 0$

by Theorem 1.

Lemma 6. C: projective $[n,k,d]_3$ code

a_i : the number of i-hps in $\Sigma =$PG$(k-1,3)$. Then

$$\sum_{i=0}^{n-d-2} \binom{n-d-i}{2} a_i$$
$$= \binom{n-d}{2}\theta_{k-1} - n(n-d-1)\theta_{k-2} \qquad (3.1)$$

θ_r denotes the number of points in PG$(r,3)$,

$$\theta_r = (3^{r+1}-1)\ /\ (3-1)$$
$$= 3^r + 3^{r-1} + \ldots + 3 + 1$$

Suppose $a_{91} > 0$, and let π be a 91-hp

the spectrum of π : $(\tau_{10}, \tau_{28}, \tau_{31}) = (1, 30, 90)$

Let c_j be the number of j-hps($\neq \pi$) through a fixed t-solid in π, then

$$\sum_j ((n-d)-j)c_j = 95 - 3t \qquad (3.2)$$

for $i = 91$ by Lemma 4.

The solution of (3.2) maximizing LHS of (3.1):

- $t = 10 : (c_{68}, c_{74}, c_{99}) = (1, 1, 1)$
- $t = 28 : (c_{95}, c_{99}, c_{101}) = (1, 1, 1)$
- $t = 31 : (c_{100}, c_{102}) = (1, 2)$

Estimating (3.1) of Lemma 6,

$$2262 \leq 942 \cdot 1 + 24 \cdot 30 + 1 \cdot 90 + 55 = 1807$$

a contradiction. Hence $a_{91} = 0$ in Σ

$\therefore \Phi_1 = a_{100} > 0$.

Let Π be a 100-hp in Σ.

100-hp $\rightarrow$ $[100, 5, 66]_3$ code (by Lemma 1).

$\Rightarrow$ The spectrum of Π is one of the following :

$$\text{(a) } (\tau_{19}, \tau_{28}, \tau_{31}, \tau_{34}) = (1, 3, 27, 90)$$

$$\text{(b) } (\tau_{25}, \tau_{28}, \tau_{31}, \tau_{34}) = (4, 1, 24, 92)$$

Let P be the point in the dual space Σ^* of Σ corresponding to Π.

Assume that the 100-hp Π has spectrum

(a) $(\tau_{19}, \tau_{28}, \tau_{31}, \tau_{34}) = (1, 3, 27, 90)$

Since C is not extendable,

the diversity (Φ_0, Φ_1) of C satisfies

$$(\Phi_0, \Phi_1) \in \{(121, 81), (94, 135), (121, 108),$$
$$(112, 126), (130, 117), (121, 135), (148, 108)\}$$

by Theorem 2.

The spectrum of Π : $(\tau_{19}, \tau_{28}, \tau_{31}, \tau_{34}) = (1, 3, 27, 90)$

Let c_j be the number of j-hps ($\neq \Pi$) through a fixed t-solid. Then by Lemma 4

$$\sum_j ((n-d)-j)c_j = 104 - 3t \qquad (3.3)$$

The solution of (3.3) maximizing LHS of (3.1):

- $t = 19$: $(c_{68}, c_{92}, c_{99}) = (1, 1, 1)$
- $t = 28$: $(c_{86}, c_{99}, c_{101}) = (1, 1, 1)$
- $t = 31$: $(c_{92}, c_{101}, c_{102}) = (1, 1, 1)$

The spectrum of Π : $(\tau_{19}, \tau_{28}, \tau_{31}, \tau_{34}) = (1, 3, 27, 90)$

Let c_j be the number of j-hps ($\neq$ Π) through a fixed t-solid. Then by Lemma 4

$$\sum_j ((n-d)-j)c_j = 104 - 3t \qquad (3.3)$$

· $t = 34$: The solutions of (3.3) are only two :

$(c_{100}, c_{102}) = (1, 2)$

$(c_{101}, c_{102}) = (2, 1)$

The spectrum of Π : $(\tau_{19}, \tau_{28}, \tau_{31}, \tau_{34}) = (1, 3, 27, 90)$

Let c_j be the number of j-hps ($\neq$ Π) through a fixed t-solid. Then by Lemma 4

$$\sum_j ((n-d)-j)c_j = 104 - 3t \qquad (3.3)$$

· $t = 34$: The solutions of (3.3) are only two :

$(c_{100}, c_{102}) = (1, 2) \rightarrow$ (0, 2)-line

$(c_{101}, c_{102}) = (2, 1) \rightarrow$ (2, 1)-line

Lemma 6. (Yoshida-Maruta, 2009)

The number of (i,j)-lines through $P \in F_1$ in a 5-flat is the following :

	$(1,3)$-line	$(0,2)$-line	$(2,1)$-line
$(121,81)_5$	13	54	54
$(94,135)_5$	40	54	27
$(121,108)_5$	31	45	45
$(112,126)_5$	40	45	36
$(130,117)_5$	40	36	45
$(121,135)_5$	49	36	36
$(148,108)_5$	40	27	54

The spectrum of Π : $(\tau_{19}, \tau_{28}, \tau_{31}, \tau_{34}) = (1, 3, 27, \underline{90})$

· $t = 34$: The solutions of (3.3) are only two :

$(c_{100}, c_{102}) = (1, 2) \rightarrow$ (0, 2)-line

$(c_{101}, c_{102}) = (2, 1) \rightarrow$ (2, 1)-line

Hence the number of 34-solids contained in two 100-hps and two 102-hps is at most 54.

Estimating (3.1) under this condition,

$$2262 \leq 609 \cdot 1 + 123 \cdot 3 + 45 \cdot 27 + 1 \cdot \underline{54} + 0 \cdot \underline{36} + 1 = 2248$$

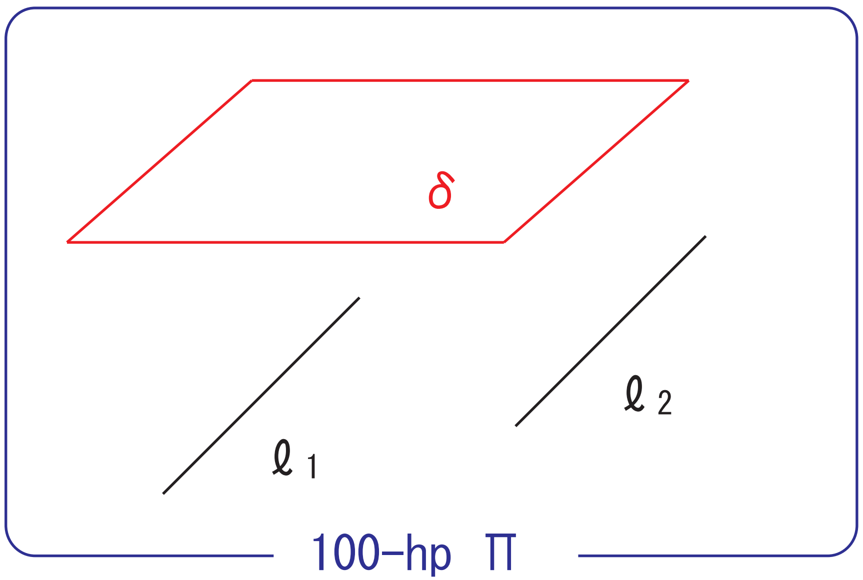
a contradiction.

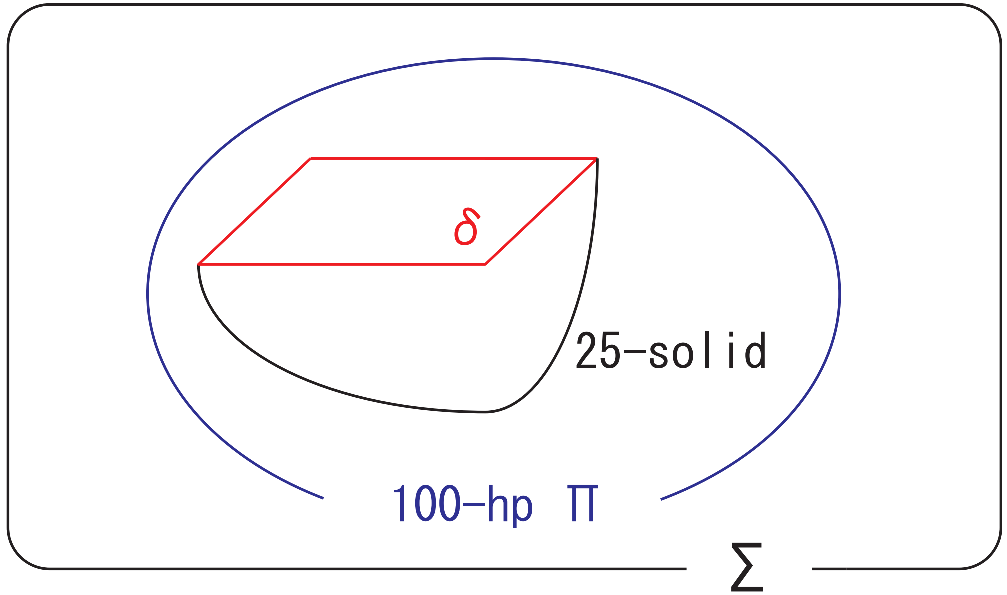
∴ There exists no 100-hp with spectrum (a).

Assume that the 100-hp Π has spectrum

(b) $(\tau_{25}, \tau_{28}, \tau_{31}, \tau_{34}) = (4, 1, 24, 92)$.

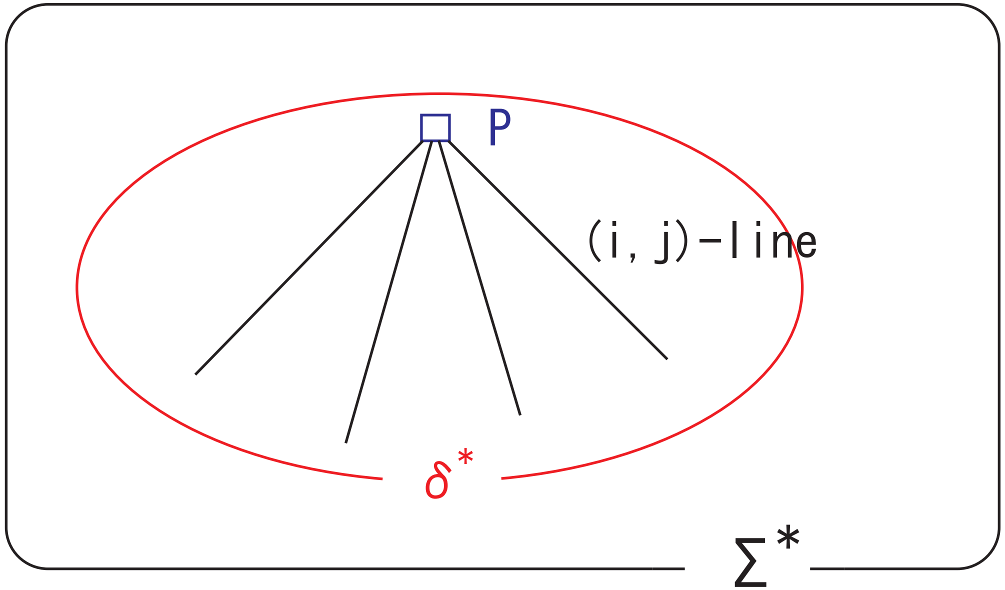
Then, $\Pi \cap C_0$ is a disjoint union of a plane δ and two lines ℓ_1, ℓ_2.





The solids containing δ in Π are 25-solids. $(\tau_{25} = 4)$

Assume that $P \in F_1$ and the (s, t)-plane δ^* in Σ^* correspond to Π and δ in Σ, respectively.



The lines through P in δ^* correspond to the 25-solids containing δ in Π.

→ What are the lines corresponding to the 25-solids?

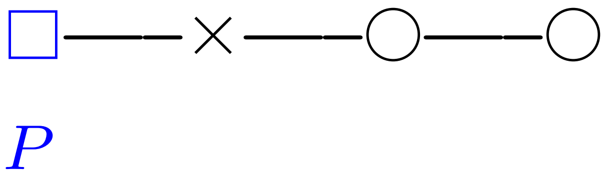
→ What is the (s, t)-plane δ^*?

Δ: a 25-solid in Π, $c_j = \#$ of j-hp $(\neq \Pi) \supset \Delta$

Since $c_{80} = c_{81} = c_{87} = c_{90} = 0$, Lemma 4 gives

$$
\begin{aligned}
&28c_{74} + 16c_{86} + 13c_{89} + 10c_{92} + 7c_{95} \\
&\qquad + 6c_{96} + 4c_{98} + 3c_{99} + 2c_{100} + c_{101} = 29 \qquad (3.4)
\end{aligned}
$$

which has no solution with $c_{100} > 0$. Since $\Phi_1 = a_{100}$, the lines through P in δ^* are $(2, 1)$-lines.



$(2, 1)$-line

Lemma 7. (Yoshida-Maruta, 2009)

The number of (i, j)-lines through $P \in F_1$ in a plane is the following :

	$(1,3)$-line	$(0,2)$-line	$(2,1)$-line
$(1,6)$-plane	1	3	0
$(4,3)$-plane	0	2	2
$(4,6)$-plane	2	1	1
$(7,3)$-plane	1	0	3
$(4,9)$-plane	4	0	0

There exists no plane in Σ^* containing four $(2,1)$-lines through $P \in F_1$ by Lemma 7.

$\Rightarrow$ a contradiction.

Hence $a_{100} = 0$, which contradicts $\Phi_1 > 0$.

$\therefore$ There exists no $[302,6,200]_3$ code.

Contents

4. Results

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$$\downarrow$$

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Thank you for your atention!

References

[1] N. Hamada, A characterization of some $[n, k, d; q]$-codes meeting the Griesmer bound using a minihyper in a finite projective geometry, Discrete Math. 116 (1993) 229–268.

[2] R. Hill, An extension theorem for linear codes, *Des. Codes Cryptogr.* **17** (1999), 151–157.

[3] T.Maruta, Extendability of ternary linear codes, Des. Codes Cryptogr. **35** (2005) 175-190.

[4] Y. Yoshida, T. Maruta, Ternary linear codes and quadrics, Electronic J. Combinatorics **16** (2009), #R9 (pp. 1-21).

[5] N. Hamada, The nonexistence of $[303, 6, 201; 3]$-codes meeting the Griesmer bound, Technical Report OWUAM-009(1995), Osaka Women's University.