(b_1, b_2)-Optimal Byte Correcting Codes

Vinod Tyagi[1] and Amita Sethi[2]

1. Department of Mathematics, SLC (E), University of Delhi, Delhi, India

e-mail:vinodtyagi@hotmail.com

2. Research Scholar, Department of Mathematics, University of Delhi, Delhi,India

e-mail:amita_sethi_23@indiatimes.com

Paper presented at the international conference on Algebraic & Combinatorial Coding Theory, Sept. 5-11, 2010, Akademgorodok, Novosibirsk, Russia

Main Idea

Byte oriented perfect codes are generally constructed for correcting any given length of a burst. We also know that codes which attain sphere packing bound are called perfect. Here we generate optimal binary linear codes organized in bytes which can correct a fixed burst of length b_1 in one byte or a burst of length b_2 (fixed) in the remaining bytes. The codes are optimal in a sense that they can correct only a given pattern of errors and no other error. In an (n, k) linear code, if there are m bytes of length β, then $n = m\beta$.

Why Byte correcting codes

In most memory and storage system, the information is stored in bytes. In such byte oriented memories, whenever an error occurs, it is generally in the form of a burst. Thus error correction in such system means correcting all errors that occur in the same byte. Therefore we consider byte correcting codes for such storage system.

Most of the studies under byte correcting codes is with respect to the usual definition of burst according to which

'A burst of length b is a vector whose all the non- zero components are confined to some b consecutive positions, the first and the last of which is nonzero'.

Five types of byte correcting codes

Our work is inspired by the work of Tuvi Etzion on perfect codes and byte oriented codes. Tuvi [6] has defined five types of byte-correcting perfect codes according to different sizes of the bytes viz.

Type 1	All bytes have the same size
Type 2	One byte of size n_1 and other bytes of size n_2
Type 3	Each byte is of either size n_1 or size n_2
Type 4	The size of each byte is a power of 2.
Type 5	All the other cases

CTD burst

There is another definition of burst due to Chien and Tang [4] with a modification due to Dass [1], named as CTD burst. According to this definition,

'*A CTD burst of length b is a vector whose all the non-zero components are confined to some b-consecutive positions, the first of which is non-zero and the number of its starting positions is* $(n - b + 1)$'.

We also call such a burst as a **burst of length b (fixed)**. It may be noted that according to this definition, (1000000) will be considered as a burst of length up to 7, whereas (0001000) will be a burst of length at most 4.

Objective

Our objective is to construct byte correcting codes with different sizes of bytes as considered by Tuvi along with different sizes of bursts. Here we consider optimal codes where all bytes have the same size but they can correct different CTD burst in different bytes.

This situation is possible in codes where it is known that a particular type of error may occur within a specified number of bytes and if one desires to increase a byte in the block length, it is natural to expect some more errors among the additional digits. However, the errors which are likely to occur in the additional digits need not necessarily be of the type existing in earlier bytes.

So there is a need to study optimal (b_1, b_2) type burst correcting optimal codes. We would be interested to find when such optimal byte correcting codes exist and when they can not.

Tyagi and Sethi bound

Tyagi and Sethi [7] proved a lower bound over the number of party check digits required for a code that corrects different fixed burst of length b_1, b_2, and b_3 in first n_1, next n_2 and last n_3 digits, $n_1+n_2+n_3=n$. The bound is based on the fact that the number of cosets is at least as large as the number of error patterns to be corrected and is given as

$$q^r \geq 1 + \sum_{i=1}^{3} (n_i - b_i + 1)(q-1)q^{b_i - 1} \qquad (1)$$

For binary case and for byte oriented codes with all bytes of the same size β and redundancy r, the bound turns out to be

$$q^r \geq 1 + \sum_{i=1}^{3} (\beta - b_i + 1)q^{b_i - 1} \qquad (2)$$

A linear code which satisfies (2) with equality is called optimal. This gives

$$q^r = 1 + \sum_{i=1}^{3} (\beta - b_i + 1) q^{b_i - 1} \qquad (3)$$

Generalization of lower bound

The result in (3) can also be generalized to m bytes correcting atmost m different fixed bursts of length b_i, i = 1, 2,......,m and can be stated as

$$q^r = 1 + \sum_{i=1}^{m} (\beta - b_i + 1) 2^{b_i - 1}$$

As we have mentioned earlier, we are interested in (b_1, b_2)-optimal burst correcting codes within bytes of size β.

(b_1, b_2)-Codes

We now construct $(m\beta, m\beta - r)$ optimal byte oriented codes for $b_1 = 1$ and $b_2 = 2$ (fixed) and for each $r \geq 3$. Conclusion and a list of open problems are suggested in the end. Let us look at two simple examples before formulating necessary and sufficient condition for the existence of such codes.

A trivial example is an optimal **(1, 2)-(6, 3)** code which corrects all single errors in the first byte of length β=3 and all burst of length 2(fixed) in the second byte. Its parity check matrix may be written as

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Other simple example is an optimal **(1, 2)-(12, 8)** burst correcting code which corrects all burst of length $b_1 = 1$ in the first byte of length β=3 and all burst of length $b_2 = 2$ (fix) in the remaining $m - 1 = 3$ bytes. Its parity check matrix may be written as

$$\mathbf{H} = \begin{bmatrix} 110 & 100 & 001 & 010 \\ 111 & 010 & 000 & 100 \\ 111 & 000 & 100 & 110 \\ 100 & 001 & 010 & 101 \end{bmatrix}$$

It can be verified that this code correct all single errors in the **first byte** and all burst of length 2 (fixed) in the **remaining three bytes**.

Necessary Condition

Now, we first give a necessary condition for the existence of such (1, 2)-optimal burst correcting codes with bytes of size β. The number of different bursts of length 2 (fixed) within a byte is 2β-2 and β is the number of single errors in a byte. The total number of nonzero vectors of length r is $2^r - 1$. Hence the total number of bytes in such a code with redundancy r is

$$\left\lceil \frac{2^r - 1}{2\beta - 2} \right\rceil + 1 \qquad (5)$$

if the **remainder R** for $\frac{2^r - 1}{2\beta - 2}$ is equal to β. In the example given above for

(12, 8) burst correcting code, for r = 4 and β = 3 we have

$$m = \left\lceil \frac{15}{4} \right\rceil + 1 = 4.$$ (Remainder = 3)

where m is the total number of bytes. It can be verified that in this case, for any other size of β>3, (1, 2) optimal byte oriented codes do not exist. So condition (5) is also sufficient.

Examples

Another interesting example is with **redundancy r = 5**. Applying necessary condition in this case, we note that the ratio $\frac{2^5 - 1}{2\beta - 2} = \frac{31}{2\beta - 2}$ has remainder **5** only when β = 3 and other when β = 7. This shows the existence of **(24, 9) and (21, 16) byte oriented codes with number of bytes m = 8 and m = 3.**

The parity-check matrix of these codes may be written as

When β = 3, m = 8

$$
H = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1
\end{pmatrix} \oplus
$$

When $\beta = 7$, $m = 3$

$$
H = \begin{pmatrix}
1\,0 \quad 1\,0 \quad 0\,0\,0 \quad 1\,0\,1\,0\,1\,1\,0 \quad 1\,0\,0\,0\,0\,1\,0 \\
0\,1 \quad 0\,1 \quad 0\,0\,0 \quad 0\,1\,1\,1\,0\,1\,0 \quad 0\,1\,0\,1\,0\,1\,1 \\
0\,0 \quad 0\,0 \quad 1\,0\,0 \quad 1\,0\,1\,0\,0\,1\,0 \quad 1\,0\,1\,1\,1\,0\,1 \\
1\,0 \quad 0\,0 \quad 0\,1\,0 \quad 1\,0\,1\,1\,0\,0\,0 \quad 0\,1\,0\,1\,1\,0\,1 \\
0\,1 \quad 0\,0 \quad 0\,0\,1 \quad 0\,1\,0\,1\,1\,0\,1 \quad 0\,0\,1\,0\,1\,1\,1
\end{pmatrix}
$$

Note that two or more different form of byte oriented codes can exist for any given

$(m\beta, m\beta - r)$ optimal code.

Construction of parity check matrix

Let us assume that $\left\lceil \frac{2^r - 1}{2\beta - 2} \right\rceil + 1 = m$ and there exists (1, 2) burst correcting code of length $m\beta$ with m bytes of size β and redundancy r.

We will describe now how to construct the parity check matrix of such (1, 2)-($m\beta$, $m\beta$-r) optimal code with m bytes of size β and redundancy r.

Let H be the parity check matrix of (1, 2) optimal ($m\beta$, $m\beta$–r) burst correcting code with bytes of size β. The columns of H = $[h_1, h_2, \ldots, h_n]$ are the elements of GF(2^r). There will always exist a vector j = $(a_1, a_2, \ldots, a_n)$, $a_i \in$ GF(2^r) such that the sums of at most two adjacent elements within m–1 bytes of size β, whose number is (2β–2) (m–1), along with all single elements in one byte of size β give each of the nonzero element of GF(2^r) exactly once. Alternatively, the jth column h_j can be added to the parity check matrix H if

1. h_j is different from all preceding j–1 columns in the first byte.

2. h_j is different from all preceding j–1 columns and also different from the sum of any two adjacent columns.

The sum of 1 and 2 comes out to be $(2\beta-2)(m-1)+\beta$. Thus the jth column h_j can always be added provided 2^r is greater than this sum. Thus we obtain the following result:

Theorem: There will always exist an $(m\beta, m\beta-r)$ linear byte oriented code that corrects all bursts of length $b_1 = 1$ in the first byte and all bursts of length $b_2 = 2$ (fixed) in the remaining n–1 bytes of size β, $m\beta = n$, satisfying the inequality

$$2^r \geq 3 + 2m\beta - 2m - \beta.$$

Our Contribution

We could give a construction for optimal $(m\beta, m\beta-r)$ (1, 2) burst correcting byte oriented codes for $r \geq 3$. We have also given the existence of (12, 8) byte correcting (1, 2) optimal code with bytes of size $\beta = 3$ and redundancy $r = 4$; (24, 19), (21, 16) byte correcting (1, 2) optimal code with bytes of size $\beta = 3$ and 7 and redundancy $r = 5$ with the help of examples.

Open Problems

. We conjecture that

1. There exist an infinite class of (1, 2) optimal byte oriented $(m\beta, m\beta-r)$ codes for redundancy $r \geq 3$.
2. Byte oriented codes can now be classified w.r.t. size of the byte as well as length of the burst.

Some other interesting questions are related to the existence of different categories of (b_1, b_2)- type byte oriented optimal codes viz.

1. If there exist other (b_1, b_2) optimal codes for $b_1 \neq 1$ and $b_2 \neq 2$ for $r \geq 3$; and
2. If there exist such (b_1, b_2) non-binary codes also.

(b_1, b_2) perfect code w.r.t. usual definition of burst

In a byte oriented code, the total number of bytes in an $(m\beta, m\beta\text{-}r)$ code with redundancy r **is**

$2^{r}-1 / 2\beta-1$. For a (b_1, b_2) perfect byte oriented code we need to calculate number of bursts of length b_1 or less and number of bursts of length b_2 or less in the byte of size β.

For example, the matrix

$$H = \begin{pmatrix} 0001 & 1111 \\ 0010 & 0011 \\ 0100 & 0101 \\ 1000 & 1111 \end{pmatrix}$$

may be considered as a parity check matrix for (3,1)-(8,4) byte oriented code where β=4. Here 4β-5 is the total number of burst of length 3or less and β is the number of single errors in a byte.

Similarly, if we consider (15,10) code with byte size β=3, then it is easy to verify that the code is a (3,2) perfect byte correcting code where 3 bytes can correct burst of length 3 or less and remaining two bytes will correct burst of length 2 or less. The existence of (9, 5) - (3, 2, 1) perfect byte correcting code with bytes of size 3 can also be proved but we have not been able to construct examples in both the cases discussed above

References

[1] B.K. Dass (1980), On a burst error correcting code, J.inf. Optimiz. Sci.,1,No.3,pp.291-295

[2] C.L. Chen, (1983),Error Correcting Codes with byte error detection capabilities, *IEEE Trans.Comput*.,vol.C-32,pp615-621

[3] C.L. Chen (1986), Byte oriented error correcting codes with semiconductor memory systems, *IEEE Trans.Comput*.,vol.C-35,pp.646-648

[4] R.T. Chien and D.T.Tang (July1965), On definition of a burst, *IBM J.Res.Develop.*, 9:292-293

[5] T.R.N. Rao and E. Fujiwara (1989), *Error control coding for computer system,* Englewood Cliff, NJ: Prentice Hall

[6] Tuvi Etzion (1998),Perfect byte correcting codes, *IEEE Trans.Inform.Theory,*vol.44,pp.3140-3146.

[7] Vinod Tyagi and Amita Sethi (2007), $(n_{1b_1}, n_{2b_2}, n_{3b_3})$ Burst Correcting Linear Codes', Accepted for publication in *Beykent University Journal of Pure and Applied Sciences (Turkey)*

THANK YOU