Local Distribution and Reconstruction of Hypercube Eigenfunctions

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Hypercube: $F^n = \{\mathbf{x} = (x_1, x_2, \dots, x_n) \ : \ \forall i \ x_i \in \{0, 1\}\}$

Hamming distance: $\rho(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i|$

Hamming weight: $wt(\mathbf{x}) = \rho(\mathbf{x}, \mathbf{0})$

The h-th level of F^n: $W_h = \{\mathbf{x} \in F^n \ : \ wt(\mathbf{x}) = h\}, \quad h = 0, 1, \dots, n$

The k-dimensional *face* γ of F^n: the set of all vertices with fixed $n-k$ coordinates.

The faces γ and $\gamma^\perp$ are *orthogonal* if the set of positions which are fixed for all vertices of $\gamma^\perp$ and the set of positions which are free for all vertices of γ coincide.

I. Introduction

We study eigenfunctions of an n-dimensional hypercube $\mathbf{F}^n$, i.e. eigenfunctions of the adjacency matrix of the hypercube graph.

The eigenvalues: $n-2i,\ i=0,1,\ldots,n.$

The eigenfunctions: $\sum_{\mathbf{y}\in N(\mathbf{x})} f(\mathbf{y}) = (n-2i)f(\mathbf{x}), \quad i=0,1,\ldots,n.$
here $\mathbf{x}\in F^n$ and $N(\mathbf{x})$ is the set of all neighbors of $\mathbf{x}$.

Note that an arbitrary equitable 2-partition (or perfect 2-coloring) can be represented by an eigenfunction which take on two different values.

The orthogonal basis of a space of all real functions over the hypercube:

$$\left\{f^{\mathbf{a}} : \mathbf{F}^n \to \mathbf{R} \;:\; f^{\mathbf{a}}(\mathbf{x}) = (-1)^{\langle \mathbf{a},\mathbf{x}\rangle}, \quad \mathbf{a} \in \mathbf{F}^n\right\}.$$

The function $f^{\mathbf{a}}$ is the eigenfunction with the eigenvalue $n - 2wt(\mathbf{a})$. So, the set of functions

$$\{f^{\mathbf{a}} : \mathbf{F}^n \to \mathbf{R} \;:\; \mathbf{a} \in W_i\}$$

forms the basis of the eigensubspace F_i with the eigenvalue $\lambda = n - 2i,\ i = 0, 1, \ldots, n$. This subspace consists of all functions such that their Fourier coefficients can be nonzero only on the i-th level of the hypercube.

The eigenfunction with the eigenvalue λ is referred to as *λ-function*.

Let γ be a k-dimensional face and $\mathbf{a} \in \gamma$. Denote

$$A_i^{\mathbf{a}}(\gamma) = \{\mathbf{x} \in \gamma \ : \ \rho(\mathbf{a}, \mathbf{x}) = i\},$$

$$v_i^{\mathbf{a}}(\gamma) = \sum_{\mathbf{x} \in A_i^{\mathbf{a}}(\gamma)} f(\mathbf{a}).$$

The *local distribution* of a function f *in the face* γ *with respect to a vertex* $\mathbf{a} \in \gamma$:

$$v^{\mathbf{a}}(\gamma) = (v_0^{\mathbf{a}}(\gamma), v_1^{\mathbf{a}}(\gamma), \ldots, v_k^{\mathbf{a}}(\gamma)).$$

The generating function of the local distribution:

$$g_\gamma^{\mathbf{a}}(t) = \sum_{i=0}^{k} v_i^{\mathbf{a}}(\gamma) t^i$$

The distribution of a code can be defined as the distribution of its characteristic function.

Let γ and $\gamma^\perp$ be orthogonal, $\gamma \bigcap \gamma^\perp = \mathbf{a}$. Denote

$$w_{ij}^{\mathbf{a}}(\gamma) = \sum_{\mathbf{x} \in F^n \;:\; \rho(\mathbf{x},\gamma)=i,\; \rho(\mathbf{x},\gamma^\perp)=j} f(\mathbf{x})$$

The *complete local distribution* of function f with respect to the face γ and vertex $\mathbf{a} \in \gamma$:

$$w^{\mathbf{a}}(\gamma) = \left(w_{ij}^{\mathbf{a}}(\gamma) \;:\; i = 0, 1, \ldots, n - \dim(\gamma),\; j = 0, 1, \ldots, \dim(\gamma) \right).$$

Theorem 1. Let $f : \mathbf{F}^n \to \mathbf{R}$ be the λ-function and γ and $\gamma^\perp$ are orthogonal faces with the common vertex $\mathbf{a}$. Then the local distribution $v^{\mathbf{a}}(\gamma)$ uniquely determines the complete local distribution $w^{\mathbf{a}}(\gamma)$ and, in particular, the local distribution $v^{\mathbf{a}}(\gamma^\perp)$.

Derive the interdependence formula of the distributions in two orthogonal faces.

Theorem 2. Let $f : \mathbf{F}^n \to \mathbf{R}$ be a λ-function and γ and $\gamma^\perp$ be orthogonal faces with the common vertex $\mathbf{a}$. Then the distribution $v^{\mathbf{a}}(\gamma^\perp)$ is uniquely determined by the distribution $v^{\mathbf{a}}(\gamma)$. Moreover, corresponding generating functions satisfy the following equation:

$$g^{\mathbf{a}}_{\gamma^\perp}(t) = (1-t)^{\frac{n-\lambda}{2}-k}(1+t)^{\frac{n+\lambda}{2}-k} g^{\mathbf{a}}_{\gamma}(-t)$$

Let

$$l(\lambda) = \min\left\{\frac{n-\lambda}{2}, \frac{n+\lambda}{2}\right\}.$$

If $k \le l(\lambda)$ then the function $(1-t)^{\frac{n-\lambda}{2}-k}(1+t)^{\frac{n+\lambda}{2}-k}$ is the polynomial of the degree $n-2k$ and then the coefficients of $g^{\mathbf{a}}_{\gamma^\perp}(t)$ are presented as finite sums of $g^{\mathbf{a}}_{\gamma}(t)$ coefficients. In this case derive an explicit formula for the distribution in the orthogonal face.

Theorem 3. Let $k \le l(\lambda)$. Then

$$v_i^{\mathbf{a}}(\gamma^\perp) = \sum_{j=0}^{k} (-1)^j P_{i-j}\left(\frac{n-\lambda}{2} - k; n-2k\right) v_j^{\mathbf{a}}(\gamma),$$

where $P_k(x;N) = \sum_{j=0}^{k}(-1)^j \binom{x}{j}\binom{N-x}{k-j}$ – Krawtchouk polynomial.

We try to reconstruct all values of a λ-function in a ball of radius h by its values in the corresponding sphere of radius h.

Theorem 4. Let $h = l(\lambda)$ and $\varphi_{\mathbf{a}}$, $\mathbf{a} \in W_h$, be arbitrary constants. Let f be an λ-function such that $f(\mathbf{a}) = \varphi_{\mathbf{a}}$, $\mathbf{a} \in W_h$. If

$$P_{h-k}\left(\frac{n-\lambda}{2} - k; n-2k\right) \neq 0, \quad k = 0, 1, \ldots, h,$$

then the whole λ-function f is uniquely determined.

Theorem 5. Let $h \leq l(\lambda)$ and $\varphi_{\mathbf{a}}$, $\mathbf{a} \in W_h$, be arbitrary constants. Let f be an λ-function such that $f(\mathbf{a}) = \varphi_{\mathbf{a}}$, $\mathbf{a} \in W_h$. If

$$P_{h-k}\left(\frac{n-\lambda}{2} - k; n-2k\right) \neq 0, \quad k = 0, 1, \ldots, h,$$

then the all values of f at the vertices of weight at most h are uniquely determined.

Theorem 6. Let $l(\lambda) \leq h \leq n/2$ and $\varphi_{\mathbf{a}},\ \mathbf{a} \in W_h$, be arbitrary constants. If there exists a λ-function f such that

$$f(\mathbf{a}) = \varphi_{\mathbf{a}},\ \mathbf{a} \in W_h, \qquad \text{and}$$

$$P_{h-k}\left(\frac{n-\lambda}{2} - k; n-2k\right) \neq 0, \quad k = 0, 1, \ldots, l(\lambda)$$

then the all values of λ-function f at the vertices of weight at most $l(\lambda)$ are uniquely determined. If furthermore

$$P_{\frac{n-\lambda}{2}-k}\left(\frac{n-\lambda}{2} - k; n-2k\right) \neq 0, \quad k = 0, 1, \ldots, l(\lambda),$$

then the whole λ-function f is uniquely determined.

S.V.Avgustinovich, A.Yu.Vasil'eva, Reconstruction of centered function by its values at two middle levels of hypercube, Discrete Analysis and Operation Research V. 10. No. 2. 2003. P. 3-16. (in Russian).

D.S. Krotov, On weight distribution of perfect structures, arXiv:0907.0001v1 [math.CO] 30 June 2009

O. Heden, On the reconstruction of perfect codes, Discrete Mathematics, V. 256. 2002. P. 479-485.

A. Yu. Vasil'eva, Local spectra of perfect binary codes, Discrete Applied Mathematics, vol. 135, n.1-3 pp. 301-307, 2004 (Translated from: A.Yu. Vasil'eva, Local spectra of perfect binary codes, Discrete analysis and operation research. 1999. V.6, No.1, 16-25)

A.Yu. Vasil'eva, On reconstruction of generalized centered functions, Proc. of Ninth Int. Workshop on Algebraic and Combinatorial Coding Theory 19-25 June, 2004, Kranevo, Bulgaria. P. 384-389.

Thank you for your attention!