Cyclically 5-edge connected graphs, fullerenes and Pogorelov polytopes

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In this talk, we discuss fruitful connections between classical and recent results of the graph theory, the polytope theory, hyperbolic geometry and algebraic topology.

A 3-valent planar 3-connected graph is *cyclically* 5*-edge connected* ($c5$*-connected*) if it has at least 5 vertices and no two circuits can be separated by cutting fewer than 5 edges. A graph is *strongly cyclically* 5*-edge connected* (c^*5*-connected*) if in addition any separation of the graph by cutting 5 edges leaves one component that is a simple circuit of 5 edges. These notions are well-known and play an important role in the graph theory. By the result of G. D. Birkhoff (1913), the famous Four Colour Theorem for planar graphs can be reduced to the class of c^*5-connected graphs. In 1974 D. Barnette and J. W. Butler shown independently that any $c5$-connected graph can be obtained from the graph of dodecahedron by a simple set of operations. An analogous description for c^*5-connected graphs was found by D. Barnette in 1977. Later a part of this result was rediscovered by T. Inoue (2008) in the context of hyperbolic geometry.

There is a remarkable geometric characterisation of $c5$-connected graphs due to A. V. Pogorelov (1967) and E. M. Andreev (1970): a combinatorial 3-polytope can be realised in Lobachevsky space as a bounded polytope with right dihedral angles if and only if its graph is $c5$-connected. We refer to such combinatorial polytopes as *Pogorelov polytopes* ($\mathcal{P}$*-polytopes*). Generalising the classical construction of Löbell (1931), A. Yu. Vesnin in 1987 described a way to produce a hyperbolic 3-manifold from any Pogorelov polytope by endowing it with an additional structure related to the hyperbolic reflection group (this structure consists of $\mathbb{Z}/2$-vectors assigned to the facets of the polytope). An important example of this additional structure arises from the Four Colour Theorem. A. Yu. Vesnin also conjectured that hyperbolic manifolds arising from 4-colourings of one special series of Pogorelov polytopes (the so-called *Löbell polytopes* or *barrels*) are isometric if and only the 4-colourings are equivalent. In 2017 V. M. Buchstaber, N. Yu. Erokhovets, M. Masuda, T. E. Panov and S. Park proved that hyperbolic manifolds arising from any Pogorelov polytopes are isometric if and only if the polytopes with additional structures are combinatorially equivalent. Using this result V. M. Buchstaber and T. E. Panov proved that hyperbolic manifolds arising from 4-colourings of any Pogorelov polytopes are isometric if and only if the colourings are equivalent, thereby verifying Vesnin's conjecture.

According to T. Doslic (2003), the class of $\mathcal{P}$-polytopes contains fullerenes, i.e. simple 3-polytopes with only 5- and 6-gonal faces. V. M. Buchstaber and N. Yu. Erokhovets (2017) obtained the results describing the class of $\mathcal{P}$-polytopes constructively:

(1) Any $\mathcal{P}$-polytope except for the k-barrels can be obtained from the 5 or the 6-barrel by a sequence of two-edges-truncations and connected sums with 5-barrels along 5-gons.

(2) Any fullerene except for the 5-barrel and the $(5, 0)$-nanotubes can be obtained from the 6-barrel by a sequence of $(2, 6; 5, 5)$-, $(2, 6; 5, 6)$-, $(2, 7; 5, 6)$-, $(2, 7; 5, 5)$-truncations such that all intermediate polytopes are either fullerenes or $\mathcal{P}$-polytopes with facets 5-, 6- and at most one additional 7-gon adjacent to a 5-gon.