The smallest eigenvalues of Hamming, Johnson and other graphs

Sebastian M. Cioabă
University of Delaware, USA
cioaba(at)udel.edu

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If S is a subset of vertices of a graph $G = (V, E)$, then $e(S, \overline{S})$ denotes the number of edges of G with exactly one endpoint in S. The MAX CUT problem is the problem of determining $mc(G) := \max_{S\subset V} e(S, \overline{S})$ over all proper subsets S of V which is the same as finding a bipartite subgraph of G with the maximum number of edges. This is a well known NP-hard problem, but in a 1995 breakthrough work, Goemans and Williamson used semidefinite programming techniques to obtain an approximation algorithm for MAX CUT with performance ratio at least $\alpha = \frac{2}{\pi}\min_{0<\theta\leq\pi}\frac{\theta}{1-\cos\theta}$ $(0.85856 < \alpha < 0.85857)$. Their work uses a semidefinite relaxation of the inequality $mc(G) \leq n\mu_{max}(G)/4$, where $\mu_{max}(G)$ equals the largest eigenvalue of the Laplacian matrix of G. If G is d-regular, this inequality becomes $mc(G) \leq n(d - \lambda_{min})/4$.

In 1999, Karloff proved that the performance ratio of the Goemans-Williamson algorithm is exactly α and showed that it is impossible to add valid linear constraints to improve this performance ratio. Karloff's work hinged on determining the smallest eigenvalue of graphs in the Johnson association scheme. Karloff determined the smallest eigenvalue of these graphs in a certain range of parameters and made a conjecture about a larger set of parameters where his result should hold. The Johnson graph $J(n, d, j)$ is the graph whose vertices are the d-subsets of a set with n elements where two vertices are adjacent when they meet in a $(d - j)$-set. Delsarte showed that the eigenvalues of the $J(n, d, j)$ are given by the Eberlein polynomials $E_j(i) = \sum_{h=0}^{i}(-1)^{i-h}\binom{i}{h}\binom{d-h}{j}\binom{n-d-i+h}{n-d-j}$ for $0 \leq i \leq d$.

Conjecture 1. [Karloff 1999] *Let $n = 2d$ and $j > d/2$. Then the smallest eigenvalue of $J(n, d, j)$ is $E_j(1)$.*

In 2000, Alon and Sudakov applied the relation between $mc(G)$ and λ_{min} and extended Karloff's results. Their work relied heavily on determining the smallest eigenvalue of certain graphs in the Hamming association scheme. In 2016, Van Dam and Sotirov used semidefinite programming techniques to study the MAX k-CUT program (maximizing the number of edges in a k-partite subgraph of a given graph) and also investigated the smallest eigenvalue of the Hamming graphs. Let $q \geq 2, d \geq 1$ be integers. Let Q be a set of size q. The Hamming scheme $H(d, q)$ is the association scheme with vertex set Q^d, and as relation the Hamming distance. The $d+1$ relation graphs $H(d, q, j)$, where $0 \leq j \leq d$, have vertex set Q^d, and two vectors of length d are adjacent when they differ in j places. Delsarte showed that the eigenvalues of $H(d, q, j)$ are given by the Krawtchouk polynomials $K_j(i) = \sum_{h=0}^{j}(-1)^h(q-1)^{j-h}\binom{i}{h}\binom{d-i}{j-h}$.

Conjecture 2. [Van Dam and Sotirov 2016] *Let $q \geq 2$ and $j \geq d - \frac{d-1}{q}$ where j is even when $q = 2$. Then the smallest eigenvalue of $H(d, q, j)$ is $K_j(1)$.*

Van Dam and Sotirov verified this conjecture by computer for all pairs (d, q) with $d \leq 30$ and $q \leq 15$. Alon and Sudakov proved the above conjecture for $q = 2, d$ large and j/d fixed. Unbeknownst to these authors, in 2013, Dumer and Kapralova proved the Van Dam-Sotirov conjecture for $q = 2$ and all d.

In this talk, I will describe these connections between the smallest eigenvalue and the max-cut of a graph and our proofs of the Karloff and Van Dam-Sotirov conjectures. Time permitting, I will discuss similar problems for other distance-regular graphs.