Construction of fullerenes and Pogorelov polytopes

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This is joint work with Victor Buchstaber

A Pogorelov polytope (*Pog*-polytope) is a combinatorial simple convex 3-polytope that has a bounded right-angled realization in the Lobachevsky (hyperbolic) space $\mathbb{L}^3$. Theorems by A. V. Pogorelov (1967) and E. M. Andreev (1970) imply that such polytopes are characterised by the condition that they are different from the 3-simplex and have no 3- and 4-belts, where a k-belt is a cyclic sequence of facets with empty common intersection such that facets are adjacent if and only if they follow each other. This condition appeared in the work by G. D. Birkhoff (1913), who proved that the 4 Colors Conjecture could be proved only for this class of polytopes and even for a smaller class with an additional restriction that any 5-belt surrounds a facet (Pog^*-polytopes). In graph theory graphs of Pog and Pog^*-polytopes are known as planar cyclically and strongly cyclically 5-edge-connected ($c5$- and c^*5-connected). In 1974 D. Barnette and J. W. Butler proved that any $c5$-connected planar graph is obtained from the graph of the dodecahedron by a sequence of operations of 3 types: an addition of an edge (A_1), a subdivision of a pentagon (A_2), and an addition of a pair of edges (A_3). A k-barrel B_k is a 3-polytope with the surface glued from two disks consisting of a k-gon surrounded by 5-gons. B_k is obtained from B_{k-1} by A_3. In 1977 D. Barnette proved that any c^*5-connected planar graph is obtained from the graph of some B_k, $k \geq 5$, by a sequence of operations A_1, and any $c5$-connected planar graph is obtained from some B_k, $k \geq 5$, by a sequence of operations A_1 and A_2. There is a special case of A_1 when we cut off two adjacent edges of a polytope by one plane (a $(2,k)$-truncation, where k is the number of edges of a face spanned by the two edges). Combining methods by D. Barnette and the authors we obtain.

Theorem 1. [1,2] *A simple 3-polytope P is Pog if and only if either $P = B_k$, $k \geq 5$, or P is obtained from B_5 or B_6 by a sequence of $(2,k)$-truncations, $k \geq 6$, and operations A_2. It is a Pog^*-polytope if and only if either $P = B_k$, $k \geq 5$, or P is obtained from B_6 by a sequence of $(2,k)$-truncations, $k \geq 6$.*

Results by T. Dǒslić (1998, 2003) imply that any fullerene (a simple 3-polytope with only 5- and 6-gonal faces) is a Pog-polytope. Results by F. Kardoš, F. Škrekovski and K. Kutnar, D. Marušič (2008) imply that a fullerene is not Pog^* if and only if it is a $(5,0)$-nanotube (i.e. is obtained from B_5 by a sequence of A_3 applied to a 5-gon surrounded by 5-gons). Denote by $\mathcal{F}_{5,\leq 7}$ the set of all simple 3-polytopes with 5-, 6- and at most one 7-gon, where the 7-gon is adjacent to a 5-gon. These polytopes are Pog ([1]).

Theorem 2. [1] *Any fullerene different from a $(5,0)$-nanotube can be obtained from B_6 by a sequence of $(2,6)$ and $(2,7)$-truncations such that intermediate polytopes belong to $\mathcal{F}_{5,\leq 7}$.*

Theorem 3. [2] *A polytope in $\mathcal{F}_{5,\leq 7}$ is not Pog^* iff $P \neq B_5$ and P contains a 5-gon surrounded by 5-gons. Such a polytope is obtained from a fullerene by a sequence of A_2. Any other polytope $P \in \mathcal{F}_{5,\leq 7}$ is obtained from B_6 by a sequence of $(2,6)$- and $(2,7)$-truncations, and operations O_1,O_2,O_3, where O_i are certain compositions of these truncations, such that intermediate polytopes also belong to $\mathcal{F}_{5,\leq 7}$.*

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References

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[2] N. Yu. Erokhovets, Construction of Fullerenes and Pogorelov Polytopes with 5-, 6- and one 7-Gonal Face. *Symmetry* **10(3)** (2018) 67.