Minimum supports of eigenfunctions in bilinear forms graphs

Ev Sotnikova
Sobolev Institute of Mathematics, Novosibirsk, Russia
lucernavesper(at)gmail.com

Bilinear forms graph $Bil_q(n,m)$ is a distance-regular graph with the vertex set V consisting of all $n\times m$ matrices over a finite field F_q and two vertices being adjacent when their matrix difference has a rank 1. A function $f : V \to \mathbb{R}$ that is not constantly zero and satisfies the equation $\theta f(u) = \sum\limits_{v\sim u} f(v) \quad \forall u \in V$ is called an *eigenfunction* of a graph corresponding to an eigenvalue θ of its adjacency matrix.

We are interested in finding eigensupports of minimum cardinality and describing their structure. This problem was explored for several families of graphs in the works [1–6]. In case of distance-regular graphs there is a well known *weight distribution* lower bound (see [1], for example) for the support cardinality which can be calculated from the intersection numbers of a graph.

This work studies the eigenfunctions of $Bil_q(n,m)$ corresponding to its minimal eigenvalue. The first part is dedicated to the case $n = m = 2$ over the prime field F_p. We prove that the weight distribution bound can be achieved and provide an explicit construction that gives rise to a desired eigenfunction with minimum support. This construction is described below:

Theorem 1. *Let a_1 be a generating element of the multiplicative group F_p^*. Denote $a_0 = 0$; $a_2 = a_1^2$; $\dots$; $a_{p-2} = a_1^{p-2}$; $a_{p-1} = a_1^{p-1} = 1$. Choose $\delta \in F_p$, such that $\delta \neq -\xi^2$ for all $\xi \in F_p$. The independent set $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} \frac{1}{a_i^2\delta+1} & \frac{a_i}{a_i^2\delta+1} \\ \frac{a_i\delta}{a_i^2\delta+1} & \frac{a_i^2\delta}{a_i^2\delta+1} \end{bmatrix}$ together with the vertices $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} \frac{1}{a_i^2\delta+1} & \frac{-a_i}{a_i^2\delta+1} \\ \frac{a_i\delta}{a_i^2\delta+1} & \frac{1}{a_i^2\delta+1} \end{bmatrix}$, where $i = 0 \dots p-1$, form a minimum eigensupport as two parts of a complete bipartite graph $K_{p+1,p+1}$.*

In the second part of the work we are recalling a well known representation of a bilinear forms graph $Bil(n,m)$ as a subgraph of a Grassmann graph $J_q(n+m,m)$. For a Grassmann graph the minimum supports are characterized and can be presented in terms of quadtratic forms and its totally isotropic spaces [7]. Using this important connection we explore the non-existence of eigenfunctions with minimum supports achieving the weight distribution bound in the case of bilinear forms graphs with a diameter at least 3.

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