BACKGROUND NOTES ON GROUP THEORY

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These notes provide some background on Group Theory for my lectures, especially for lecture 7 on Hurwitz Groups and Surfaces.

1. Notation

Here is some standard group theory notation:

- C_n, a cyclic group of order $n \in \mathbb{N} \cup \{\infty\}$.
- D_n, a dihedral group of order $2n$, $n \in \mathbb{N} \cup \{\infty\}$ (some write D_{2n} here).
- S_n, the symmetric group of degree n, $n \in \mathbb{N}$.
- A_n, the alternating group of degree n, $n \in \mathbb{N}$.
- F_n, the free group of rank $n \in \mathbb{N} \cup \{\infty\}$.
- $G = \langle X \mid R \rangle$, the group with generating set X and defining relations R.
- $G_1 * G_2$, the free product $\langle X_1 \cup X_2 \mid R_1 \cup R_2 \rangle$ of two groups $G_i = \langle X_i \mid R_i \rangle$.
- $|G : H|$, the index of a subgroup H of G, the number of its cosets in G.
- $[a, b]$, the commutator $a^{-1}b^{-1}ab$ of a and b.
- G', the commutator subgroup or derived group of G, generated by its commutators.
- h^g, the conjugate $g^{-1}hg$ of an element h by g.
- H^g, the conjugate $g^{-1}Hg$ of a subgroup H by g.
- $N_G(H)$, the normaliser $\{g \in G \mid H^g = H\}$ of a subgroup H in G.
- $C_G(h)$, the centraliser $\{g \in G \mid h^g = h\}$ of an element h in G.

2. Linear and projective groups

For any field F, the *general linear group* $GL_n(F)$ is the group of all $n \times n$ invertible matrices with entries in F, and the *special linear group* $SL_n(F)$ is the normal subgroup consisting of those matrices of determinant 1. Taking $n = 2$, the centre of $SL_2(F)$ consists of the matrices $\pm I$, and the quotient group $SL_2(F)/\{\pm I\}$ is the *projective special linear group* $PSL_2(F)$, sometimes abbreviated to $L_2(F)$. Its elements are the pairs $\pm M$ of matrices $M \in SL_2(F)$, and it is isomorphic to the group of Möbius transformations

$$t \mapsto \frac{at+b}{ct+d}, \quad a, b, c, d \in F, \quad ad - bc = 1$$

of $\mathbb{P}^1(F) = F \cup \{\infty\}$. It can be proved that $PSL_2(F)$ is a simple group for any field F with $|F| > 3$.

Similarly $PGL_2(F)$, the quotient of $GL_2(F)$ by its centre $\{\lambda I \mid \lambda \in F \setminus \{0\}\}$, is the *projective general linear group*, isomorphic to the group of Möbius transformations

$$t \mapsto \frac{at+b}{ct+d}, \quad a, b, c, d \in F, \quad ad - bc \neq 0$$

of $\mathbb{P}^1(F)$. We have $PGL_2(F) = PSL_2(F)$ if each element of $F \setminus \{0\}$ is a square in F; otherwise $PSL_2(F)$ is a subgroup of index 2 in $PGL_2(F)$.

If F is the finite field $\mathbb{F}_q = GF(q)$ of order q (necessarily a prime power), then the groups $GL_2(F)$, etc, are sometimes denoted by $GL_2(q)$, and so on. By counting linearly independent rows we see that

$$|GL_2(q)| = (q^2 - 1)(q^2 - q) = q(q - 1)^2(q + 1).$$

The determinant map $GL_2(q) \to \mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$ is an epimorphism with kernel $SL_2(q)$, so

$$|SL_2(q)| = |GL_2(q)|/(q - 1) = q(q^2 - 1).$$

If $q = 2^e$ then $-I = I$ and so $PSL_2(q) = SL_2(q)$, also of order $q(q^2 - 1)$. However, if q is odd then

$$|PSL_2(q)| = |SL_2(q)|/2 = q(q^2 - 1)/2.$$

For example, $|PSL_2(7)| = 168$.

Some useful isomorphisms: $PSL_2(2) \cong S_3 \cong D_3$, $PSL_2(3) \cong A_4$, $PGL_2(3) \cong S_4$, $PSL_2(4) \cong PSL_2(5) \cong A_5$, $PGL_2(5) \cong S_5$, $PSL_2(7) \cong GL_3(2) = PGL_3(2)$, $PSL_2(9) \cong A_6$.

3. Sylow's Theorems

These state that if G is a finite group of order $n = p^e m$, where $e \geq 1$ and p is a prime not dividing m, then

- G has subgroups of order p^e (called Sylow p-subgroups);
- the Sylow p-subgroups are all conjugate to each other;
- the number n_p of Sylow p-subgroups divides n and satisfies $n_p \equiv 1 \bmod (p)$;
- every subgroup of order a power of p is contained in a Sylow p-subgroup.

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