Combinatorial curvature for infinite planar graphs

Bobo Hua
Fudan University, Shanghai, China
bobohua(at)fudan.edu.cn

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In this talk, we consider infinite planar tessellations, which are planar graphs embedded in $\mathbb{R}^2$ satisfy some regularity properties for cellular structures. We call them planar graphs for simplicity. Given a planar graph $G = (V, E, F)$, we endow its ambient space $\mathbb{R}^2$ with a canonical piecewise flat metric by identifying its faces with regular Euclidean polygons and gluing them along common edges. The metric space is called the polyhedral surface $S(G)$ for the planar graph G. The generalized Gaussian curvature in the sense of Alexandrov of the polyhedral surface $S(G)$ concentrates on the vertices, which was given by the angle defect at vertices. The combinatorial curvature of a planar graph, which was introduced by [6,7] etc., is defined as $\Phi(x) = \frac{1}{2\pi}K(x)$, where $K(x)$ is the generalized Gaussian curvature at the vertex x.

We are interested in infinite planar graphs with nonnegative combinatorial curvatures, which are equivalent to that their polyhedral surfaces are generalized convex surfaces in the sense of Alexandrov. The total curvature of an infinite planar graph $G = (V, E, F)$ is defined as $\Phi(G) := \sum_{x\in V} \Phi(x)$, whenever it makes sense. The Cohn-Vossen type theorem, proven by [2], yields that for any infinite planar graph G with nonnegative combinatorial curvature $\Phi(G) \leq 1$. In this talk, we [4] will show that for any infinite planar graph G with nonnegative combinatorial curvature,

$$\Phi(G) = \frac{k}{12}, \quad k = 0, 1, \cdots, 12.$$

It was proved by [1] that for an infinite planar graph with nonnegative combinatorial curvature there are only finitely many vertices with positive curvature. We [5] prove that except the class of prism-like graphs, the number of vertices with positive curvature in an infinite planar graph with nonnegative combinatorial curvature is at most 132, see [3] for the results on planar graphs with positive curvature. As an application, the (cellular) automorphism group of an infinite planar graph with nonnegative combinatorial curvature and positive total curvature is finite.

References

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