Coincidence of Gruenberg–Kegel graphs of non-isomorphic finite groups

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Let G be a finite group. Denote by $\pi(G)$ the set of all prime divisors of the order of G and by $\omega(G)$ the *spectrum* of G, i.e. the set of all its element orders. The set $\omega(G)$ defines the *Gruenberg–Kegel graph* (or the *prime graph*) $\Gamma(G)$ of G; in this simple graph the vertex set is $\pi(G)$, and distinct vertices p and q are adjacent if and only if $pq \in \omega(G)$. Denote the number of connected components of $\Gamma(G)$ by $s(G)$, and the set of connected components of $\Gamma(G)$ by $\{\pi_i(G) \mid 1 \leq i \leq s(G)\}$; for a group G of even order we assume that $2 \in \pi_1(G)$. The following problem is of interest.

Problem. *Describe cases when Gruenberg–Kegel graphs of non-isomorphic finite groups coincide.*

Firstly, in the present talk we give a survey of some existing results.

Secondary, we concentrate on the following partial case.

Recall, a finite group G is a *Frobenius group* if there is a non-trivial subgroup C of G such that $C \cap gCg^{-1} = \{1\}$ whenever $g \notin C$. A subgroup C is a *Frobenius complement* of G. Let

$$K = \{1\} \cup (G \setminus \bigcup_{g \in G} gCg^{-1}).$$

Then K is a normal subgroup of a Frobenius group G with a Frobenius complement C. Note that K is called the *Frobenius core* of G.

Recall, the *socle* $Soc(G)$ of a finite group G is the subgroup of G generated by the set of all its non-trivial minimal normal subgroups. A finite group G is *almost simple* if $Soc(G)$ is a finite nonabelian simple group. It is well-known that a finite group G is almost simple if and only if there exists a finite nonabelian simple group S such that $S \cong Inn(S) \trianglelefteq G \leq \mathrm{Aut}(S)$.

Gruenberg–Kegel Theorem. *If G is a finite group with disconnected Gruenberg–Kegel graph, then one of the following statements holds*:

(1) *G is a Frobenius group*;

(2) *G is a 2-Frobenius group, i. e., $G = ABC$, where A and AB are normal subgroups of G, AB and BC are Frobenius groups with cores A and B and complements B and C, respectively*;

(3) *G is an extension of a nilpotent $\pi_1(G)$-group by a group A, where $S \trianglelefteq A \leq \mathrm{Aut}(S)$, S is a finite nonabelian simple group with $s(G) \leq s(S)$, and A/S is a $\pi_1(G)$-group.*

The cases when Gruenberg–Kegel graphs of a finite nonabelian simple group and of a group from item (1) or (2) of the Gruenberg-Kegel theorem coincide were described in [1]. The cases when Gruenberg–Kegel graphs of a finite almost simple group and of a solvable group from item (1) or a group from item (2) of the Gruenberg-Kegel theorem coincide can be found with using of [2].

In the present talk we consider the cases when Gruenberg–Kegel graphs of a finite almost simple group and of a nonsolvable group from item (1) of the Gruenberg-Kegel theorem coincide.

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References

[1] M. R. Zinov'eva, V. D. Mazurov, On finite groups with disconnected prime graph. *Proc. Steklov Inst. Math.* **283**:S1 (2013) 139–145.

[2] I. B. Gorshkov, N. V. Maslova, Finite almost simple groups whose Gruenberg–Kegel graphs are equal to the Gruenberg–Kegel graphs of solvable groups. *Algebra and Logic*, to appear.