The Terwilliger algebra of a tree

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Let Γ be a finite connected simple graph. Let X denote the vertex set of Γ and $V = \bigoplus_{x \in X} \mathbb{C}x$ the standard module, i.e., the vector space for which X is an orthonormal basis. Fix a vertex $x_0 \in X$ and let X_i be the set of vertices that have distance i from x_0. Then the standard module V is decomposed into the orthogonal sum $V = \bigoplus_{i=0}^{D} V_i^*$, where $V_i^* = \bigoplus_{x \in X_i} \mathbb{C}x$. The *Terwilliger algebra* $\mathfrak{T}$ of Γ is by definition the subalgebra of $\mathrm{End}(V)$ generated by the adjacency matrix A of Γ and the orthogonal projections $E_i^* : V \to V_i^*$, $0 \leq i \leq D$. Let G be the automorphism group of Γ and H the stabilizer in G of the base vertex x_0: $G = \mathrm{Aut}(\Gamma)$, $H = G_{x_0}$. Then it is easy to see that $\mathfrak{T}$ is contained in the centralizer algebra of H, i.e., each element of $\mathfrak{T}$ commutes with the action of every element of H: $\mathfrak{T} \subseteq \mathrm{Hom}_H(V, V)$.

In this talk, we discuss the Terwilliger algebra of a tree. Precisely speaking, we assume Γ is a rooted tree with x_0 the root and we let $\mathfrak{T}$ be the Terwilliger algebra of Γ with respect to x_0. We show:
(1) $\mathfrak{T} = \mathrm{Hom}_H(V, V)$, i.e., $\mathfrak{T}$ coincides with the centralizer algebra of H.
(2) The $\mathfrak{T}$-module V determines the rooted tree Γ up to isomorphism.
In particular, $\mathfrak{T} = \mathrm{End}(V)$ holds if and only if the rooted tree Γ does not have any symmetry, i.e., $H = 1$.

This talk is based on joint work with Shuang-Dong Li, Jing Xu, Masoud Karimi and Yizheng Fan. We acknowledge that Jack Koolen conjectured: For almost all finite connected simple graphs, $\mathfrak{T} = \mathrm{End}(V)$ holds regardless the base point x_0. This conjecture motivated our study on the Terwilliger algebra of a tree.