Non-residually Finite Direct Limits of the 2-Bridge Link Groups

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This is joint work with Donghi Lee

A group G is said to be *residually finite* if for each $1 \neq g \in G$ there is a finite group H and a homomorphism $\varphi : G \longrightarrow H$ such that $\varphi(g) \neq 1$. Most of well-known finitely presented groups are residually finite, and constructions of non-residually finite groups are rare.

One of the representative open problem is whether every hyperbolic group is residually finite, and it is commonly belived that a non-residually finite hyperbolic group exists.

The relative hyperbolicity and the small cancellation conditions of finitely presented groups are closely related to hyperbolic groups. So we construct a non-Hopfian group (i.e. non-residually finite) satisfying the small cancellation conditions $C(4)$ and $T(4)$ by taking a direct limit of relatively hyperbolic groups $\{G_n \,|\, n = 0, 1, 2, \cdots\}$. Each G_n has the form $G_n = \langle a, b \,|\, u_{r_0} = u_{r_1} = u_{r_2} = \cdots = u_{r_n} = 1\rangle$, where u_{r_i} is the relator of the presentation of the 2-bridge link group of slope r_i for each $i \geq 0$. There are specific rational numbers which satisfy the above conditions.

Theorem 1. *Let $r_i = [7, i\langle 3, 1, 1\rangle, 4, 5]$ for every integer $i \geq 0$. Also for every integer $n \geq 0$, let $G_n = \langle a, b \,|\, u_{r_0} = u_{r_1} = \cdots = u_{r_n} = 1\rangle$ which is hyperbolic relative to a set $\{H_n, K_n\}$ of groups. Here, $H_n = \langle a, v_{r_0}, v_{r_1}, \ldots, v_{r_n}\rangle$ and $K_n = \langle b, w_{r_0}, w_{r_1}, \ldots, w_{r_n}\rangle$ are proper subgroups of G_n, where $(u_{r_i}) \equiv (av_{r_i}a^{-1}v_{r_i}^{-1}) \equiv (w_{r_i}bw_{r_i}^{-1}b^{-1})$ for every $i = 0, 1, \ldots, n$. Then the direct limit G of a sequence*

$$G_0 \twoheadrightarrow G_1 \twoheadrightarrow G_2 \twoheadrightarrow \cdots$$

equipped with the canonical epimorphism $\alpha_n : G_n \twoheadrightarrow G_{n+1}$ at each $n \geq 0$ is infinitely presented as $G = \langle a, b \,|\, u_{r_0} = u_{r_1} = u_{r_2} = \cdots = 1\rangle$ which satisfies small cancellation conditions $C(4)$ and $T(4)$, and is non-Hopfian.

The symbol "$i\langle 3, 1, 1\rangle$"represents i successive $(3, 1, 1)$'s. Let M be a sufficiently large integer. Then we can obtain the group $\mathfrak{G}_n := \langle a, b \,|\, a^M = b^M = u_{r_0} = u_{r_1} = \cdots = u_{r_n} = 1\rangle$ is hyperbolic for each integer $n \geq 0$.

Theorem 2. *Let $r_i = [7, i\langle 3, 1, 1\rangle, 4, 5]$ for every integer $i \geq 0$. Also for every integer $n \geq 0$, let $\mathfrak{G}_n = \langle a, b \,|\, a^M = b^M = u_{r_0} = u_{r_1} = \cdots = u_{r_n} = 1\rangle$ which is hyperbolic. Then the direct limit G of a sequence*

$$\mathfrak{G}_0 \twoheadrightarrow \mathfrak{G}_1 \twoheadrightarrow \mathfrak{G}_2 \twoheadrightarrow \cdots$$

equipped with the canonical epimorphism $\alpha_n : \mathfrak{G}_n \twoheadrightarrow \mathfrak{G}_{n+1}$ at each $n \geq 0$ is infinitely presented as $\mathfrak{G} = \langle a, b \,|\, a^M = b^M = u_{r_0} = u_{r_1} = u_{r_2} = \cdots = 1\rangle$, and is non-Hopfian.

Until Theorem 2, we could construct non-residually finite $\mathfrak{G}$ such that $\mathfrak{G}$ is still infinitely presented. In order to construct a hyperbolic group, we need to have the group finitely presented at least. In the near future (hopefully before I give a talk on G2R2 conference), we expect to construct another group by adding some generators or relators which results in deleting some relators so that the group becomes finitely presented.

References

[1] D. Lee, M. Sakuma, A family of two generator non-Hopfian groups. *International Journal of Algebra and Computation* **27** (2017) 655–675.

[2] D. Lee, M. Sakuma, Epimorphisms between 2-bridge link groups: homotopically trivial simple loops on 2-bridge spheres. *Proc. London Math. Soc.* **104** (2012) 359–386.