Lecture 1: Riemann Surfaces and Fuchsian Groups (G2S2, Novosibirsk, 2018)

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Introduction

Riemann surfaces play a central role in mathematics, since they are of interest in areas such as algebra, analysis, geometry, number theory and topology.

They also have uses in theoretical physics, such as in string theory.

Finding links between these aspects of Riemann surfaces uses ideas from graph theory and combinatorics, especially maps on surfaces.

These links, mainly with algebra and combinatorics, rather than the deeper theory of Riemann surfaces, will be my main theme.

This lecture will cover the basic concepts about Riemann surfaces.

All of the other areas (except string theory!) will appear in subsequent lectures.

Fortunately, only basic knowledge of those areas is usually required.

When more advanced ideas are needed, I will summarise them, either in the lectures or in accompanying notes.

In this lecture I will start with an informal and then a formal definition of a Riemann surface, followed by some simple examples.

The (very deep) Uniformisation Theorem implies that every Riemann surface is a quotient, by a ‘nice’ group of automorphisms, of one of three of these simple examples, namely the Riemann sphere, the complex plane and the hyperbolic plane.

In the first two cases, the quotients are easily described: the most interesting examples are tori, otherwise known as elliptic curves.

The third case is richer and more complicated, involving hyperbolic geometry rather than spherical or euclidean geometry. The ‘nice’ groups which arise here are the Fuchsian groups, discrete groups of Möbius transformations of the upper half plane.

I will finish by describing some of the basic properties of Fuchsian groups, to be used in later lectures.

Intuitive concept of a Riemann surface

Riemann surfaces were introduced (by Riemann!) to understand and explain many-valued complex functions, such as $\sqrt{z}$ and $\log z$.

They can be constructed by taking copies of the complex plane $\mathbb{C}$ (or of a subset of it), joined to each other across cuts.

The result is a surface in which every point has a neighbourhood which 'looks like' an open subset of $\mathbb{C}$.

This gives local complex coordinates, allowing the basic operations of complex analysis such as differentiation and integration.

For consistency, one requires that when neighbourhoods overlap, the coordinate transition functions should be biholomorphic (conformal with a conformal inverse).

The results are locally very similar to those in the classical case, but globally they can be very different, depending on the geometry and topology of the surface.

The formal definition of a Riemann surface

Although we will not explicitly use it (I rarely do!), it is important to know and understand the formal definition of a Riemann surface.

A Riemann surface is a connected Hausdorff topological space X such that

- every point in X has an open neighbourhood U with a homeomorphism $\phi : U \to V$ to an open subset $V \subseteq \mathbb{C}$;
- whenever two such neighbourhoods U_1 and U_2, with functions ϕ_1 and ϕ_2, satisfy $U_1 \cap U_2 \neq \emptyset$, the composition

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

  is biholomorphic.

Thus the local coordinate functions ϕ assign complex coordinates $z = \phi(x)$ to points $x \in X$; the coordinate transition functions $\phi_2 \circ \phi_1^{-1}$ force any changes of local coordinates to be conformal.

Simple examples

The easiest example of a Riemann surface is trivial: $X = \mathbb{C}$, with one neighbourhood $U = \mathbb{C}$ and the identity coordinate function.

More generally, X could be any open subset of $\mathbb{C}$, such as the unit disc $\mathbb{D}$, given by $|z| < 1$, and the upper half plane $\mathbb{H}$ ($\mathrm{Im}\, z > 0$).

The first non-trivial example is the Riemann sphere or extended complex plane or complex projective line

$$X = \Sigma = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} = \mathbb{P}^1(\mathbb{C});$$

take $U_1 = \mathbb{C}$ and $U_2 = (\mathbb{C} \setminus \{0\}) \cup \{\infty\}$, with local coordinates z and $1/z$ (defining $1/\infty = 0$), so on $U_1 \cap U_2 = \mathbb{C} \setminus \{0\}$ the coordinate transition function $z \mapsto 1/z$ is biholomorphic.

Stereographic projection identifies X with the unit sphere $S^2 \subset \mathbb{R}^3$: ∞ and 0 are the north and south poles, the unit circle S^1 is the equator, and the disc $\mathbb{D}$ is the southern hemisphere. (What is $\mathbb{H}$?)

The torus

A lattice in $\mathbb{C}$ is an additive subgroup

$$\Lambda = \Lambda(\omega_1, \omega_2) = \{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}\}$$

where the generators ω_1, ω_2 are linearly independent over $\mathbb{R}$.

The quotient group $X = \mathbb{C}/\Lambda$ is called a torus.

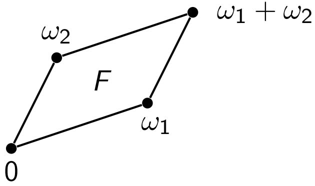
Since ω_1 and ω_2 are linearly independent, Λ is discrete: a small neighbourhood of 0 contains no other lattice points.

Near any $z_0 \in \mathbb{C}$ the projection $\mathbb{C} \to X$ is therefore one-to-one, so lifting back from X to $\mathbb{C}$ gives local coordinates near $z_0 + \Lambda \in X$; the transition functions are translations, and hence biholomorphic.

To construct the torus $\mathbb{C}/\Lambda$, take a fundamental parallelogram

$$F = \{s\omega_1 + t\omega_2 \mid s, t \in I = [0,1] \subset \mathbb{R}\}$$

and identify opposite pairs of boundary points, equivalent mod Λ: put $s\omega_1 = s\omega_1 + \omega_2$, and $t\omega_2 = \omega_1 + t\omega_2$, for all $s, t \in I$.



The translates of F by elements of Λ tessellate $\mathbb{C}$, meaning that they cover $\mathbb{C}$, overlapping only at their boundary points.

Equivalently, every $z \in \mathbb{C}$ is congruent mod Λ to a point $z' \in F$, which is unique unless it lies on the boundary ∂F of F.

Isomorphisms and automorphisms

An isomorphism or conformal equivalence $X \to X'$ of Riemann surfaces is a biholomorphic bijection.

One can show that tori $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$ are isomorphic iff the lattices Λ and Λ' are similar, that is, $\Lambda' = a\Lambda$ where $0 \neq a \in \mathbb{C}$.

The automorphisms of X are the isomorphisms $X \to X$; these form the automorphism group $\operatorname{Aut} X$. One can show that:

$\operatorname{Aut} \hat{\mathbb{C}} = PSL_2(\mathbb{C})$, the group of Möbius transformations

$$z \mapsto \frac{az+b}{cz+d} \quad (a, \ldots, d \in \mathbb{C}, ad - bc = 1);$$

$\operatorname{Aut} \mathbb{C} = AGL_1(\mathbb{C})$, the group of affine transformations

$$z \mapsto az + b \quad (a, b \in \mathbb{C}, a \neq 0).$$

Exercise Show that all rotations of S^2 are automorphisms of $\hat{\mathbb{C}}$. Find one which induces an isomorphism $\mathbb{D} \to \mathbb{H}$.

Simple connectivity

There is a very wide variety of Riemann surfaces.

However, for many purposes it is sufficient to understand just three of them, together with their automorphism groups.

We have in fact already met all three of these important surfaces.

A connected topological space X is simply connected if its fundamental group $\pi_1 X$ is trivial. Put simply, this means that every closed path in X can be continuously contracted, within X, to a point, so that there are no 'holes' in X.

For example $\mathbb{C}$ is simply connected (apply a scaling factor approaching 0), and so are $\hat{\mathbb{C}}$, $\mathbb{H}$ and $\mathbb{D}$.

However, $X = \mathbb{C} \setminus \{0\}$ is not: a closed path in X going once (or any non-zero number of times) around 0 cannot be contracted to a point without passing through 0 at some stage, and $0 \notin X$.

Uniformisation

By covering space theory, every Riemann surface X is isomorphic to $\tilde{X}/\Gamma$ where $\tilde{X}$ (its universal covering space) is a simply connected Riemann surface and Γ is a discontinuous subgroup of $\mathrm{Aut}\,\tilde{X}$.

Here 'discontinuous' means that every point in $\tilde{X}$ has an open neighbourhood N with $N \cap g(N) = \emptyset$ for all non-identity $g \in \Gamma$. In particular, such elements g have no fixed points.

The Uniformisation Theorem (Poincaré and Koebe, 1907) states that, up to isomorphism, there are just three simply connected Riemann surfaces: $\mathbb{C}$, $\hat{\mathbb{C}}$ and $\mathbb{H}$.

It follows that every Riemann surface is isomorphic to a quotient of $\mathbb{C}$, $\hat{\mathbb{C}}$ or $\mathbb{H}$ by a discontinuous group of automorphisms.

Exercise. Show that the only discontinuous subgroup of $\mathrm{Aut}\,\hat{\mathbb{C}}$ is the identity subgroup.

It follows that the only quotient of $\hat{\mathbb{C}}$ is $X \cong \hat{\mathbb{C}}$ itself.

Quotients of $\mathbb{C}$

Next we consider the case $\tilde{X} = \mathbb{C}$.

Exercise. Show that the only discontinuous subgroups of $\operatorname{Aut}\mathbb{C}$ are the identity subgroup, infinite cyclic groups generated by a translation $z \mapsto z + b$ $(b \neq 0)$ and lattices.

The first and third cases give quotients $X \cong \mathbb{C}$ and tori $\mathbb{C}/\Lambda$.

In the second case, all subgroups $\Gamma_b = \langle z \mapsto z + b\ (b \neq 0)\rangle$ are conjugate to each other (**exercise**!), so their quotient surfaces are mutually isomorphic (**another exercise**!), and we may take $b = 1$.

Identifying each $z \in \mathbb{C}$ with $z + 1$ makes $\mathbb{C}/\Gamma_1$ look like an infinite tube (take $\{z \mid 0 \leq \operatorname{Re} z \leq 1\}$ as a fundamental region).

However, the function $z \mapsto e^{2\pi i z}$, which is periodic with period 1, induces an isomorphism $\mathbb{C}/\Gamma_1 \to \mathbb{C} \setminus \{0\}$, giving $X \cong \mathbb{C} \setminus \{0\}$.

This leaves 'only' the case $\tilde{X} = \mathbb{H}$.

Automorphisms of $\mathbb{H}$

To deal with the third case $\tilde{X} = \mathbb{H}$, we need to know $\mathrm{Aut}\,\mathbb{H}$.

One can show that, as in the case of $\mathbb{C}$, the automorphisms of $\mathbb{H}$ are those of $\hat{\mathbb{C}}$ preserving its subset $\mathbb{H}$, and (**exercise!**) these form the group $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\{\pm I\}$ of Möbius transformations

$$z \mapsto \frac{az+b}{cz+d} \quad (a, \dots, d \in \mathbb{R}, ad - bc = 1).$$

We need to know the discontinuous subgroups of $PSL_2(\mathbb{R})$; however, it is convenient also to study the wider class of properly discontinuous subgroups. This condition means that every $z \in \mathbb{H}$ has an open neighbourhood N with $N \cap g(N) = \emptyset$ for all g with $g(z) \neq z$. Now, non-identity elements g may have fixed points.

One can prove that these subgroups coincide with the discrete subgroups of $PSL_2(\mathbb{R})$, called Fuchsian groups. Here the topology is that induced by embedding $SL_2(\mathbb{R})$ in the euclidean space $\mathbb{R}^4$.

The modular group

The most important Fuchsian group in the modular group

$$\Gamma = PSL_2(\mathbb{Z}) = \{z \mapsto \frac{az+b}{cz+d} \mid a, \ldots, d \in \mathbb{Z}, ad - bc = 1\}.$$

It is discrete because $\mathbb{Z}$ is a discrete subset of $\mathbb{R}$. The elements

$$X : z \mapsto \frac{-1}{z}, \quad Y : z \mapsto \frac{-1}{z-1} \quad \text{and} \quad Z : z \mapsto z+1.$$

of Γ satisfy $X^2 = Y^3 = XYZ = 1$. Row operations in $SL_2(\mathbb{Z})$ show that X and Z (and hence any two of X, Y, Z) generate Γ. We will see later that these relations are defining relations for Γ, so

$$\Gamma = \langle X, Y \mid X^2 = Y^3 = 1 \rangle \cong C_2 * C_3,$$

a free product of cyclic groups $\langle X \rangle$ and $\langle Y \rangle$ of order 2 and 3.

Exercise Show that every $z \in \mathbb{H}$ is equivalent under Γ to an element z' of the set $F = \{z \in \mathbb{H} \mid -1/2 \leq \mathrm{Re}(z) \leq 1/2, |z| \geq 1\}$. Given z, is this element z' unique?

Hyperbolic geometry

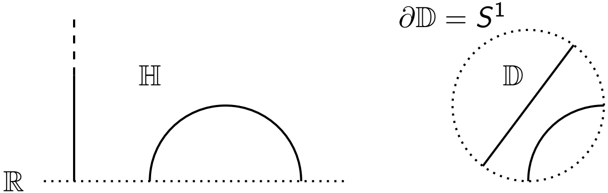
One can regard $\mathbb{H}$ as a model of the hyperbolic plane.

Length and area are obtained by integrating the elements $ds = dz/y$ and $dA = dx\,dy/y$, where $z = x + iy \in \mathbb{H}$.

$\mathbb{H}$ is a metric space, and $\mathrm{Aut}\,\mathbb{H} = PSL_2(\mathbb{R})$ is its group of orientation-preserving isometries.

Exercise Describe the orientation-reversing isometries of $\mathbb{H}$.

The geodesics in $\mathbb{H}$ and $\mathbb{D}$ are segments of euclidean lines and circles meeting the boundary at right angles.



Fixed points of automorphisms of $\mathbb{H}$

Define the trace of an element $T : z \mapsto \frac{az+b}{cz+d}$ of $PSL_2(\mathbb{R})$ to be the pair $\mathrm{tr}(T) = \pm(a+d)$ (the traces of the corresponding matrices).

Exercise. Show that a non-identity element T has two complex conjugate fixed points in $\mathbb{C}$, or one or two fixed points in the boundary $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$ of $\mathbb{H}$ as $\mathrm{tr}(T) < 2, = 2$ or > 2.

We call T an elliptic, parabolic or hyperbolic element respectively.

Exercise Show that $PSL_2(\mathbb{R})$ acts transitively on $\mathbb{H}$ and 2-transitively (transitively on distinct ordered pairs) on $\partial\mathbb{H}$.

Transitivity on $\partial\mathbb{H}$ implies that each parabolic element is conjugate to one fixing ∞; these have the form $z \mapsto z + b$ $(b \in \mathbb{R} \setminus \{0\})$.

2-transitivity on $\partial\mathbb{H}$ implies that each hyperbolic element is conjugate to one fixing 0 and ∞, of the form $z \mapsto kz$ $(k > 0)$.

For elliptic elements, it is better to work with $\mathbb{D}$; each elliptic element is conjugate to one fixing 0, a euclidean rotation about 0.

Summary

This lecture has covered some basic ideas concerning Riemann surfaces and Fuchsian groups.

The next lectures will concentrate on compact Riemann surfaces, their automorphism groups, and the maps which can be formed by embedding graphs in them.