Lecture 2: Compact Riemann Surfaces and their Automorphism Groups. (G2S2, Novosibirsk, 2018)

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Introduction

In Lecture 1 we saw that any Riemann surface X is uniformised as a quotient of one of the three simply connected Riemann surfaces $\hat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{H}$ by a discontinuous group of automorphisms.

The first two cases are easily described, so it remain to consider the hyperbolic case, where the relevant groups are all Fuchsian groups, properly discontinuous groups of automorphisms of $\mathbb{H}$.

After considering some simple examples, the elementary Fuchsian groups, I will concentrate mainly on cocompact Fuchsian groups, those with a compact quotient space. Triangle groups are useful examples of these, while the modular group is an important non-cocompact Fuchsian group.

I will show how fundamental regions yield information about presentations of finitely generated Fuchsian groups, the indices of their inclusions, and the areas and automorphism groups of their quotient spaces.

Elementary Fuchsian groups

A Fuchsian group Γ is a discrete subgroup of $PSL_2(\mathbb{R})$, equivalently a properly discontinuous group of automorphisms of $\mathbb{H}$.

The following simple examples are important because they are exceptions to some of the main theorems about Fuchsian groups.

An elliptic element $T \in PSL_2(\mathbb{R})$ is a hyperbolic rotation around a fixed point in $\mathbb{H}$; in a Fuchsian group, it must have finite order, otherwise its orbits would not be discrete, so $\Gamma = \langle T \rangle$ is finite.

A parabolic or hyperbolic element, conjugate in $PSL_2(\mathbb{R})$ to $z \mapsto z + b$ $(0 \neq b \in \mathbb{R})$ or $z \mapsto kz$ $(k > 0)$, generates an infinite cyclic group Γ. In the latter case a half-turn $U : z \mapsto -1/z$ inverts T, giving an infinite dihedral group $\Gamma = \langle T, U \rangle$.

These cyclic and dihedral groups Γ are the elementary Fuchsian groups, the only ones with a cyclic subgroup of finite index.

Cocompactness

I will concentrate mainly on cocompact Fuchsian groups, those for which the Riemann surface $X = \mathbb{H}/\Gamma$ is compact.

One can show that any such group is finitely generated, and has a fundamental region in $F \subset \mathbb{H}$ which is a polygon with finitely many sides, called a fundamental polygon.

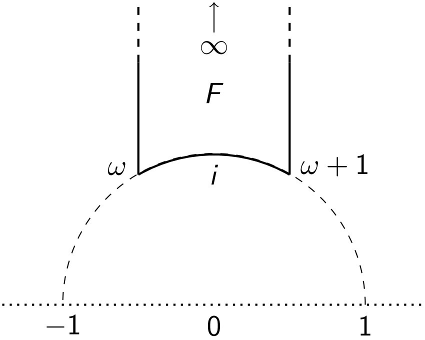
As for lattices, a fundamental region for Γ is a connected set F whose images under Γ tessellate $\mathbb{H}$; equivalently, every $z \in \mathbb{H}$ is in the same orbit as some $z' \in F$, which is unique unless it is in $\partial\mathbb{F}$. Then X can be formed by identifying equivalent points in $\partial\mathbb{F}$.

Example. The set $F = \{z \in \mathbb{H} \mid 1 \leq |z| \leq k\}$ is a fundamental region for the group $\Gamma = \langle T : z \mapsto kz \rangle$ if $k > 1$. Identifying each $z \in \partial F \cap S^1$ with kz gives an open annulus, which is non-compact.

Exercise Find fundamental regions for the other elementary Fuchsian groups. Are these groups cocompact?

Fundamental region for the modular group

The region $|\mathrm{Re}(z)| \leq 1/2, |z| \geq 1$ is a fundamental region for the modular group $\Gamma = PSL_2(\mathbb{Z})$. It is a triangle with vertices $\omega, \omega + 1$ and ∞, but not compact since $\infty \notin \mathbb{H}$. Thus Γ is not cocompact. (We will see later that $\mathbb{H}/\Gamma \cong \mathbb{C}$, which confirms this.)



Exercise Draw the images of F under the generators X, Y and Z for Γ, and persuade yourself that its images under Γ tessellate $\mathbb{H}$.

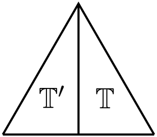
Triangle groups

If $\mathbb{T}$ is a triangle in $\mathbb{H}$ with internal angles π/l, π/m and π/n for some integers $l, m, n \geq 2$, then reflections in the sides of $\mathbb{T}$ generate a discrete group of isometries of $\mathbb{H}$. This is not a Fuchsian group, but its orientation-preserving subgroup of index 2, the triangle group $\Gamma = \Delta(l, m, n)$, is a Fuchsian group.

It is generated by rotations through $2\pi/l$, $2\pi/m$ and $2\pi/n$ (each a product of two reflections) around the vertices of $\mathbb{T}$.

The union of $\mathbb{T}$ with an adjacent reflected image $\mathbb{T}'$ is a fundamental polygon F for Γ. Identifying equivalent sides of F gives a topological sphere, which is compact, so Γ is cocompact.

Here is an analogous euclidean example in $\mathbb{C}$, with $l, m, n = 2, 3, 6$:



Poincaré's method

Poincaré discovered a method for obtaining a presentation, in terms of generators and relations, for a finitely generated Fuchsian Γ, from a fundamental polygon for Γ.

The generators are side-pairing elements, automorphisms of $\mathbb{H}$ which carry one side to F to another (or to itself in some cases, by means of a half-turn about its midpoint).

The defining relations are obtained by a more complicated process, which involves going round the boundary ∂F of F and looking at how F meets its neighbours at the vertices.

By making group-theoretic transformations (equivalently, by rearranging F), this presentation can be put into a standard form, which gives useful information about Γ and $X = \mathbb{H}/\Gamma$.

The standard presentation

A finitely generated Fuchsian group Γ has generators

$$A_i, B_i \quad (i = 1, \ldots, g) \quad (\textit{hyperbolic})$$

$$X_j \quad (j = 1, \ldots, r) \quad (\textit{elliptic})$$

$$Y_k \quad (k = 1, \ldots, s) \quad (\textit{parabolic})$$

$$Z_l \quad (l = 1, \ldots, t) \quad (\textit{hyperbolic})$$

with defining relations

$$X_j^{m_j} = 1 \quad (j = 1, \ldots, r)$$

for integers $m_j \geq 2$, together with the 'long relation'

$$\prod_i [A_i, B_i] \prod_j X_j \prod_k Y_k \prod_l Z_l = 1,$$

where $[A_i, B_i]$ denotes the commutator $A_i^{-1} B_i^{-1} A_i B_i$.

The signature

The standard presentation of Γ can be encoded in its signature

$$(g; m_1, \ldots, m_r; s; t),$$

giving the number of generators (or pairs of them) of each type, and the elliptic periods, the orders m_j of the elliptic generators X_j. The signature gives information about the surface $X = \mathbb{H}/\Gamma$:

- The genus of X is g.
- The periods m_j correspond to cone-points of order m_j in X, where the total angle around a point is $2\pi/m_j$; these arise from side identifications at vertices of F, as in triangle groups.
- s is the number of punctures in X, arising from ideal vertices of F on $\partial\mathbb{H}$, as in $PSL_2(\mathbb{Z})$ at ∞.
- t is the number of holes in X, where F meets $\partial\mathbb{H}$ along a line-segment, as in the elementary Fuchsian groups.

Examples

The modular group $\Gamma = PSL_2(\mathbb{Z})$ has signature $(0; 2, 3; 1; 0)$; X is a sphere with cone-points of order 2 and 3 and a puncture.

A triangle group $\Gamma = \Delta(l, m, n)$ has signature $(0; l, m, n; 0; 0)$; X is a sphere with cone-points of order l, m and n.

The elementary Fuchsian group Γ generated by a hyperbolic element has signature $(0; -; 0; 2)$; X is a sphere with two holes.

Exercise Find the signatures of the other types of elementary Fuchsian groups, and check that their presentations are correct.

A surface group Γ, uniformising a compact Riemann surface X of genus $g \geq 2$, has signature $(g; -; 0; 0)$ and a presentation

$$\langle A_i, B_i \ (i = 1, \ldots, g) \mid \prod_i [A_i, B_i] = 1 \rangle.$$

It is isomorphic to the fundamental group $\pi_1 X$ of X.

Hyperbolic area

$X = \mathbb{H}/\Gamma$ inherits a hyperbolic geometry from $\mathbb{H}$, and hence it has a hyperbolic area $\mu(X) = \mu(F)$, where F is a fundamental region.

If F is a fundamental polygon, its area can be found by dividing F into triangles, applying the Gauss–Bonnet formula to each of them, and adding their areas.

The Gauss–Bonnet formula, that a triangle with internal angles α_i has area $\pi - \sum_i \alpha_i$, can be proved by moving the triangle by an isometry so that one side is along the imaginary axis, and then integrating $dx\,dy/y$ over the triangle.

The result is that if Γ has signature $(g; m_1, \ldots, m_r; s; t)$, then $\mu(X)$ is finite if and only if $t = 0$ (no holes!), in which case

$$\mu(X) = \mu(F) = 2\pi \left(2g - 2 + \sum_{j=1}^{r} \left(1 - \frac{1}{m_j} \right) + s \right).$$

Examples

The modular group $\Gamma = PSL_2(\mathbb{Z})$ has signature $(0; 2, 3; 1; 0)$, so

$$\mu(X) = 2\pi \left(-2 + (1 - \frac{1}{2}) + (1 - \frac{1}{3}) + 1 \right) = \frac{\pi}{3}.$$

A triangle group $\Gamma = \Delta(l, m, n)$ has signature $(0; l, m, n; 0; 0)$, so

$$\mu(X) = 2\pi \left(1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n} \right).$$

The elementary Fuchsian group Γ generated by a hyperbolic element has signature $(0; -; 0; 2)$; here $t = 2 \neq 0$, so $\mu(X) = \infty$. The same result applies to the other elementary Fuchsian groups.

A surface group Γ, uniformising a compact Riemann surface $X \cong \mathbb{H}/\Gamma$ of genus $g \geq 2$, has signature $(g; -; 0; 0)$, so

$$\mu(X) = 2\pi(2g - 2).$$

The Riemann–Hurwitz formula

If K is a subgroup of finite index $N = |\Gamma : K|$ in Γ, then a fundamental region F' for K can be formed from N images $T(F)$ of a fundamental region F for Γ, one for each of a suitably chosen set of coset representatives T for K in Γ.

Thus if $\mu(F)$ is finite, then so is $\mu(F')$, with $\mu(F') = N\mu(F)$. This proves the Riemann–Hurwitz Formula, that in these circumstances

$$\mu(\mathbb{H}/K) = |\Gamma : K|\mu(\mathbb{H}/\Gamma).$$

In practical examples, one usually knows some or all of the parameters in the signatures of Γ and K.

Example If a surface group K of genus g is a subgroup of finite index in a triangle group $\Gamma = \Delta(l, m, n)$, then

$$2g - 2 = |\Gamma : K| \left(1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}\right).$$

Automorphism groups of Riemann surfaces

If K is a subgroup of a group G, its normaliser in G is

$$N_G(K) = \{g \in G \mid K^g := g^{-1}Kg = K\},$$

the largest subgroup in which K is normal. Clearly $K \leq N_G(K)$.
For a subgroup $K \leq PSL_2(\mathbb{R})$, let us write $N_{PSL_2(\mathbb{R})}(K)$ as $N(K)$.

Theorem
If $X = \mathbb{H}/K$ where K is a torsion-free Fuchsian group (one with no elliptic elements), then $\mathrm{Aut}\, X \cong N(K)/K$.

Theorem
If K is a non-elementary Fuchsian group then $N(K)$ is Fuchsian group.

We saw in Lecture 1 that any compact Riemann surface X of genus $g \geq 2$ is uniformised by a discontinuous (and hence torsion-free) Fuchsian surface group K which is not elementary (since they are not cocompact), so these theorem apply.

In particular, $|\mathrm{Aut}\, X| = |N(K) : K|$, and since K and hence $N(K)$ are cocompact, the Riemann–Hurwitz formula

$$2\pi(2g-2) = \mu(K) = |N(K) : K|\mu(N(K))$$

applies, giving

$$|\mathrm{Aut}\, X| = \frac{2\pi(2g-2)}{\mu(N(K))}.$$

Thus information about $N(K)$ (or simply its signature) can tell us about the order of $\mathrm{Aut}\, X$. Examples will follow in later lectures.

Exercise Show that if K is any elementary Fuchsian group, then $N(K)$ is not properly discontinuous.