Lecture 3: Triangle Groups and their Quotients (G2S2, Novosibirsk, 2018)

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August 4, 2018

Introduction

It this lecture I will concentrate on Fuchsian groups of the simplest possible signature, namely $(0; m_1, m_2, m_3)$ (s, t omitted if both 0).

They are called triangle groups because they are generated by rotations around the vertices of triangles in $\mathbb{H}$.

They are important for us because they (and their analogues in spherical and euclidean geometry) will be used later to describe maps and hypermaps on surfaces.

Triangle groups are also important for their applications in other areas such a Number Theory, Analysis and Geometry.

I will briefly mention some of these connections, as they are of general interest, even though they will not be used later.

Extended triangle groups

Let $\mathbb{T}$ be a triangle with internal angles π/m_i at its vertices A_i $(i = 1, 2, 3)$ for integers $m_i \geq 2$.

Such a triangle can be drawn in $\hat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{H}$ as

$$\sum_i \frac{1}{m_i} > 1, \quad = 1 \quad \text{or} \quad < 1.$$

I will assume $\mathbb{T}$ is in $\mathbb{H}$ (or $\mathbb{D}$); the other two cases are similar.

The reflection R_i in the side of $\mathbb{T}$ opposite A_i is an isometry of $\mathbb{H}$.

The extended triangle group $\Delta^* = \Delta[m_1, m_2, m_3]$ is the group of isometries generated by the reflections R_i $(i = 1, 2, 3)$.

If $i \neq j$ then $R_i R_j$ is a rotation through $2\pi/m_k$ around A_k, where $\{i, j, k\} = \{1, 2, 3\}$, so $2m_k$ images of $\mathbb{T}$ fit together around A_k.

This implies that the images of $\mathbb{T}$ tessellate a region of $\mathbb{H}$ near $\mathbb{T}$, and one can use this to show that they tessellate all of $\mathbb{H}$.

Orientation-preserving triangle groups

The orientation-preserving elements of Δ^* form a subgroup of index 2, consisting of the words of even length in the generators R_i.

This is the ordinary or orientation-preserving triangle group $\Delta = \Delta(m_1, m_2, m_3)$, generated by $X_1 = R_2R_3$, $X_2 = R_3R_1$, $X_3 = R_1R_2$.

The generators for Δ^* satisfy the relations

$$R_i^2 = (R_iR_j)^{m_k} = 1 \quad (i, j, k \text{ all distinct}).$$

One can use the simple connectedness of $\mathbb{H}$ to prove that these are defining relations for Δ^*.

It follows easily from this that Δ has defining relations

$$X_i^{m_i} = X_1X_2X_3 = 1 \quad (i = 1, 2, 3).$$

I will sometimes use the alternative triangle group notation

$$\Delta = \Delta(l, m, n) = \langle X, Y, Z \mid X^l = Y^m = Z^n = XYZ = 1 \rangle.$$

Rigidity

Of course, a triangle $\mathbb{T}$ in $\mathbb{H}$ is not uniquely determined by its internal angles, so the triangle groups Δ and Δ^* are not uniquely determined by the triples (m_i), called their type.

However, any two triangles in $\mathbb{H}$ of the same type are equivalent under some isometry (possibly reversing orientation).

This follows from the facts that (unlike in euclidean geometry) the internal angles determine the side-lengths, and $\mathrm{Aut}\,\mathbb{H}$ acts transitively on pairs of points a given distance apart.

It follows easily that any two triangle groups or extended triangle groups of the same type are conjugate under an element of $\mathrm{Aut}\,\mathbb{H}$.

By abuse of language we can therefore refer to 'the triangle group' of a given type.

This property, called rigidity, fails for all other signatures of Fuchsian groups.

Fundamental regions

Since its images under Δ^* tessellate $\mathbb{H}$, $\mathbb{T}$ is a fundamental region for Δ^*, that is, each $z \in \mathbb{H}$ is equivalent under Δ^* to some $z' \in \mathbb{T}$, which is unique unless it is in the boundary of $\mathbb{T}$.

Similarly, $F := \mathbb{T} \cup R_i(\mathbb{T})$ is a fundamental region for Δ for any i.

It follows that the orbit of any $z \in \mathbb{H}$ under either group is a discrete set, and hence that Δ is a discrete subgroup of $\mathrm{Aut}\,\mathbb{H} = PSL_2(\mathbb{R})$, that is, a Fuchsian group.

Any element T of finite order in Δ must be elliptic, with a fixed point in $\mathbb{H}$.

Exercise Hence prove that T is conjugate to a power of some X_i.

Identifying equivalent sides of F (draw it!) gives a topological sphere, so Δ is cocompact, and hence has no parabolic elements.

Thus all elements of Δ of infinite order are hyperbolic.

The signature of a subgroup

Th Riemann–Hurwitz formula states that if K is a subgroup of finite index in Δ, then

$$\mu(K) = |\Delta : \Gamma| \mu(\Delta).$$

If $\Delta = \Delta(m_1, m_2, m_3)$, with signature $(0; m_1, m_2, m_3)$, then $\mu(\Delta) = \mu(F) = 2\pi(1 - \sum_i m_i^{-1})$ (by the Gauss–Bonnet formula).

Being cocompact, K has a signature $(g; n_1, \ldots, n_r)$, with

$$\mu(K) = 2\pi\big(2g - 2 + \sum_j (1 - n_j^{-1})\big).$$

By a result of Singerman (1970), the elliptic periods n_j are the numbers m_i/l where l is the length of a 'short' cycle of X_i in the action of Δ on the cosets of K, one having length $l < m_i$.

If this action is known, the periods n_j can be found, and then g is given by the Riemann–Hurwitz formula.

Special cases

If K is a surface group, then it has no elliptic periods n_j, so $\mu(K) = 2\pi(2g-2)$ and hence

$$2g - 2 = |\Delta : K|(1 - \sum_i m_i^{-1}).$$

This can impose strong conditions on the inclusion $K \leq \Delta$.

Example If $\Delta = \Delta(2,3,7)$ then $1 - \sum_i m_i^{-1} = 1/42$, so the index $|\Delta : \Gamma|$ must be divisible by 84. (We will revisit this example later.)

Exercise Show that if K is a normal subgroup of Δ, so that Δ acts regularly as $G = \Delta/K$ on the cosets of K, then

$$2g - 2 = |\Delta : K|(1 - \sum_i l_i^{-1})$$

where l_i is the order of the image of X_i in Δ/K. What is the signature of K in this case?

Finite quotient groups

The finite quotients of triangle groups are the 2-generator finite groups, including all the finite simple groups, and many others.

Example It is known (exercise!) that the symmetric group S_n is generated by the n-cycle $(1, 2, \ldots, n)$ and the transposition $(1, 2)$.

They have product $(2, 3, \ldots, n)$, so there is an epimorphism $\theta : \Delta = \Delta(n, 2, n-1) \to S_n$.

The kernel K is a normal surface subgroup, of genus g given by

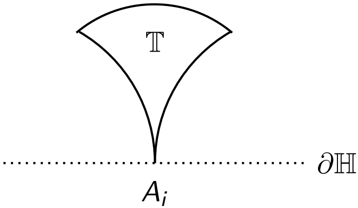
$$2g - 2 = |S_n|(1 - \frac{1}{n} - \frac{1}{2} - \frac{1}{n-1}) = \frac{n^2 - 5n + 2}{2}(n-2)!.$$

For example, taking $n = 5$ gives a Riemann surface $\mathbb{H}/K$ of genus 4 with automorphism group S_5; this is Bring's surface (Bring, 1786!).

Exercise Find the signature of $\theta^{-1}(S_{n-1})$.

Non-cocompact triangle groups

This theory can be extended to allow $\mathbb{T}$ to have internal angles equal to 0 by using ideal vertices A_i on the boundary $\partial\mathbb{H} = \hat{\mathbb{R}}$ of $\mathbb{H}$.



In this case the 'rotation' $R_j R_k$ ($\{i, j, k\} = \{1, 2, 3\}$) around A_i is parabolic rather than elliptic, of order $m_i = \infty$, and Δ is not cocompact: $\mathbb{H}/\Delta$ is a sphere with a puncture at the image of A_i, so it is not compact.

The modular group $\Delta(2,3,\infty)$

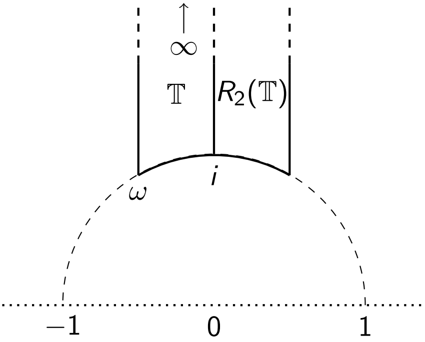
This is the most important non-cocompact triangle group.

Let $\mathbb{T}$ have internal angles $\pi/2, \pi/3$ and 0 at i, $\omega = e^{2\pi i/3}$ and ∞.

Then Δ is the modular group $PSL_2(\mathbb{Z})$, with fundamental region

$$F = \mathbb{T} \cup R_2(\mathbb{T}) = \{z \in \mathbb{H} \mid -1/2 \leq \mathrm{Re}(z) \leq 1/2, |z| \geq 1\},$$

and Δ^* is the extended modular group $PGL_2(\mathbb{Z}) = GL_2(\mathbb{Z})/\{\pm I\}$.



Presentations for the modular group

The modular group $\Gamma = PSL_2(\mathbb{Z})$ is the triangle group $\Delta(2, 3, \infty)$, so it has a presentation

$$\Gamma = \langle X, Y, Z \mid X^2 = Y^3 = XYZ = 1 \rangle$$

(we ignore 'relations' like $Z^\infty = 1$ in the case of ideal vertices).

We can eliminate the redundant generator $Z = (XY)^{-1}$ to give

$$\Gamma = \langle X, Y \mid X^2 = Y^3 = 1 \rangle.$$

Thus $\Gamma \cong C_2 * C_3$, the free product of groups $\langle X \mid X^2 = 1 \rangle \cong C_2$ and $\langle Y \mid Y^3 = 1 \rangle \cong C_3$; this is the 'largest' group generated by two elements of order 2 and 3, in the sense that every other such group is a quotient of Γ.

Exercise Show that A_5 is a quotient of Γ; is S_5 a quotient?

Congruence subgroups of the modular group

Reduction mod (n) is a ring epimorphism $\mathbb{Z} \to \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$.

It induces an epimorphism

$$\theta_n : \Gamma = PSL_2(\mathbb{Z}) \to PSL_2(\mathbb{Z}_n) = SL_2(\mathbb{Z}_n)/\{\pm I\}.$$

Exercise Why is it not obvious that θ_n is an epimorphism? When you understand this, prove that it is an epimorphism!

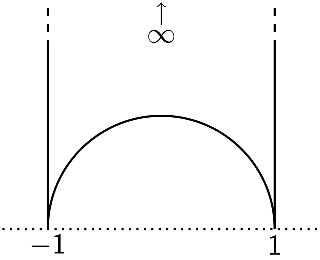
We define $\Gamma(n) = \ker(\theta_n)$, the principal congruence subgroup of level n. It consists of the elements $z \mapsto (az + b)/(cz + d)$ of Γ with

$$a \equiv d \equiv \pm 1 \quad \text{and} \quad b \equiv c \equiv 0 \bmod (n).$$

Any subgroup of Γ containing some $\Gamma(n)$ is a congruence subgroup.

Congruence subgroups are important in Number Theory. They all have finite index, but the converse is false: 'most' subgroups of finite index in Γ are noncongruence subgroups.

Exercise Show that $PSL_2(\mathbb{Z}_2) \cong S_3$, so that $|\Gamma : \Gamma(2)| = 6$.
Show that the hyperbolic triangle with ideal vertices at $1, -1$ and ∞ shown below is a fundamental region for $\Gamma(2)$.
Show that $\Gamma(2)$ is a free group of rank 2.
Is $\Gamma(2)$ the only normal subgroup of index 6 in Γ?



It can be shown that if $n > 2$ then

$$|\Gamma : \Gamma(n)| = \frac{n^3}{2} \prod_{p|n} \left(1 - \frac{1}{p^2}\right),$$

where p ranges over the distinct primes dividing n.

The modular group and tori

A pair ω_1, ω_2, linearly independent over $\mathbb{R}$, generate a lattice Λ.

Define their modulus $\tau = \omega_2/\omega_1$, transposing subscripts if necessary so that $\tau \in \mathbb{H}$.

Two such pairs generate the same lattice iff they are equivalent under $SL_2(\mathbb{Z})$, and hence generate similar lattices iff their moduli are equivalent under $\Gamma = PSL_2(\mathbb{Z})$.

Thus isomorphism classes of tori $\mathbb{C}/\Lambda$ correspond to orbits of Γ on $\mathbb{H}$, that is, points in $\mathbb{H}/\Gamma$, or equivalently F with sides identified.

The sides $\mathrm{Re}(z) = \pm 1/2$ are identified by the translation $Z : z \mapsto z + 1$, and the two halves of the side $|z| = 1$ by the half-turn $X : z \mapsto -1/z$ around i.

The square and hexagonal lattices, spanned by $1, i$ and $1, \omega$, correspond to $\tau = i$ and ω (equivalent to $\omega + 1 = -1/\omega$).

Exercise Which $\tau \in F$ correspond to real lattices, with $\overline{\Lambda} = \Lambda$?

The modular function J

For a lattice $\Lambda \subset \mathbb{C}$, define $g_2 = g_2(\Lambda) = 60 \sum'_\omega \omega^{-4}$ and $g_3 = g_3(\Lambda) = 140 \sum'_\omega \omega^{-6}$, summing over all non-zero $\omega \in \Lambda$.

One can show that

$$J := \frac{g_2^3}{g_2^3 - 27g_3^2}$$

depends only on the similarity class of Λ, so the modular function $J(\tau) = J(\Lambda)$, where Λ has basis $\{\tau, 1\}$, is invariant under Γ.

It is an analytic function $\mathbb{H} \to \mathbb{C}$, giving an isomorphism $\mathbb{H}/\Gamma \to \mathbb{C}$.

Thus $J(i) = 1$ and $J(\omega) = 0$ (the square and hexagonal lattices).

Being periodic, J has a Fourier series expansion, given by

$$J(\tau) = \frac{1}{1728}\left(\frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots\right)$$

where $q := e^{2\pi i\tau}$. Monstrous moonshine arose from connections between these coefficients and the degrees $1, 19883, 21296876, \ldots$ of the irreducible representations of the Monster simple group.