Lecture 4: Maps and Hypermaps on Surfaces (G2S2, Novosibirsk, 2018)

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Introduction

A map is an embedding of a graph in a surface.

We will see how maps on oriented surfaces can be described by using permutations.

These generate a group called the monodromy group, and the permutations commuting with them form its automorphism group.

The most symmetric maps are the orientably regular maps, for which the automorphism group acts transitively on directed edges.

We will consider chiral maps, those which form mirror-image pairs.

Finally we will generalise maps to hypermaps: these are embeddings of hypergraphs in surfaces, giving a link between Riemann surfaces and Galois theory.

Definition of a map

A map is an embedding of a graph in a surface, without crossings, and with simply connected faces (homeomorphic to discs).

For simplicity, and because all Riemann surfaces are orientable, assume that the surface is oriented (orientable with a chosen orientation).

An idea of Hamilton (1856) describes a map as a pair of permutations.

It has been rediscovered several times, with minor variations.

In all cases, the idea is to allow group-theoretic ideas to be applied to maps, which are basically topological or combinatorial objects.

Maps and permutations

Given an oriented map $\mathcal{M}$, let Φ be its set of arcs (directed edges).

Define x to be the permutation of Φ which follows the orientation around each incident vertex, so cycles of $x \longleftrightarrow$ vertices of $\mathcal{M}$.

Define y to be the permutation which reverses the direction of each arc, so cycles of $y \longleftrightarrow$ edges of $\mathcal{M}$.

Then cycles of $z := (xy)^{-1} = y^{-1}x^{-1} \longleftrightarrow$ faces of $\mathcal{M}$.)

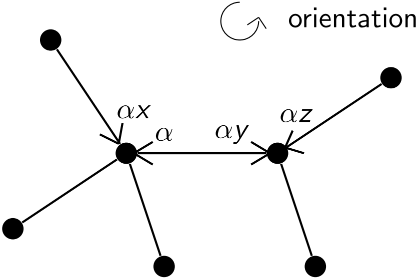


Figure: The permutations x, y and z

Clearly $y^2 = xyz = 1$, so let Δ be the triangle group

$$\langle X, Y, Z \mid Y^2 = XYZ = 1 \rangle = \langle X, Y \mid Y^2 = 1 \rangle = \Delta(\infty, 2, \infty).$$

Then $\mathcal{M}$ determines an action $\theta : \Delta \to \mathrm{Sym}\,\Phi$ (the symmetric group on Φ) of Δ on Φ by

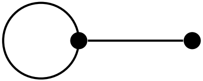
$$X \mapsto x, \quad Y \mapsto y, \quad Z \mapsto z.$$

Conversely, any permutation representation of Δ determines an oriented map $\mathcal{M}$, with the cycles of X, Y and Z as vertices, edges and faces, incidence given by non-empty intersection, and orientation given by the cyclic order within cycles.

Connected maps correspond to transitive representations, compact $(=$ finite) maps correspond to finite representations.

We will generally assume both these conditions.

A small example



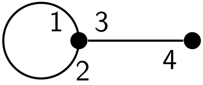


Figure: A map on the sphere

Let $\mathcal{M}$ be the map on the sphere on the left (drawn in the plane, by stereographic projection). Numbering the arcs we get

$$x = (1, 2, 3)(4) = (1, 2, 3) \quad \text{and} \quad y = (1, 2)(3, 4),$$

so that

$$z = (1)(2, 3, 4) = (2, 3, 4)$$

with cycles of length 1 and 3 corresponding to two faces of valencies 1 and 3. (The edge with arcs 3 and 4 contributes two sides to the face containing it.)

Warnings

1. The permutations x, y and z are, in general, not automorphisms of the map or the graph, since they do not preserve incidence.

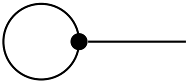
They are simply instructions on how to join arcs to form a map.

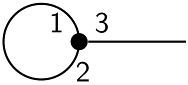
2. A fixed point of y corresponds to a free edge, or semi-edge, incident with a vertex at only one end: think of this as the quotient of an ordinary edge by the half-turn about its mid-point.

For example, if we remove the vertex of valency 1 from the preceding example, as below, there are just three arcs. Now

$$x = (1, 2, 3), \quad y = (1, 2)(3) = (1, 2), \quad \text{so} \quad z = (1)(2, 3) = (2, 3)$$

corresponding to two faces of valencies 1 and 2. (The free edge contributes only one side to the face containing it.)





Monodromy and automorphism groups

The group $G = \langle x, y \rangle \leq \mathrm{Sym}\,\Phi$ is the monodromy group of $\mathcal{M}$.

The (orientation-preserving) automorphisms of $\mathcal{M}$ are the permutations of Φ commuting with x and y, so the (orientation-preserving) automorphism group $A = \mathrm{Aut}\,\mathcal{M}$ is the centraliser $C(G)$ of G in $\mathrm{Sym}\,\Phi$.

Example/Exercise. Let $\mathcal{M}$ embed a path of n edges in the sphere, as in the diagram. Numbering the arcs as below, so that

$$x = (1)(2,3)\ldots(2n-2, 2n-1)(2n), \quad y = (1,2)(3,4)\ldots(2n-1, 2n),$$

check that xy is a $2n$-cycle, and G is a dihedral group of order $4n$. What is $C(G)$?

Lemma

Let a group G act transitively on a set Φ, and let A be its centraliser in $\mathrm{Symm}\,\Phi$. Then A acts semiregularly (fixed-point-freely) on Φ, and $A \cong N_G(G_\alpha)/G_\alpha$ for each $\alpha \in \Phi$.

(G_α is the subgroup of G fixing an arc $\alpha \in \Phi$, and $N_G(G_\alpha)$ is its normaliser $\{h \in G \mid h^{-1}G_\alpha h = G_\alpha\}$ in G.)

Hence if $\mathcal{M}$ is a (connected) map then $\mathrm{Aut}\,\mathcal{M}$ acts semiregularly on Φ, and

$$A \cong N_G(G_\alpha)/G_\alpha \cong N_\Delta(M)/M$$

for any arc $\alpha \in \Phi$, where $M = \Delta_\alpha$ (called a map subgroup of Δ) is the inverse image of G_α in Δ (both unique up to conjugation).

Exercise Taking $\alpha = 1$ in the preceding Example, show that $G_\alpha = \langle x \rangle \cong C_2$ and $N_G(G_\alpha) = \langle x, (xy)^n \rangle \cong C_2 \times C_2$, so $A \cong C_2$.

Orientably regular maps

The most symmetric oriented maps $\mathcal{M}$ are the orientably regular maps, where A acts transitively on Φ. The Lemma implies:

Lemma

If $\mathcal{M}$ is an oriented map the following are equivalent:

- *$\mathcal{M}$ is orientably regular,*
- *A is a regular permutation group,*
- *G is a regular permutation group,*
- *M is a normal subgroup of Γ.*

When these conditions are satisfied we have $A \cong G \cong \Delta/M$.

Example The platonic solids can be regarded as orientably regular maps on the sphere, with $A \cong A_4, S_4$ and A_5 for the tetrahedron, the cube and octahedron, and the dodecahedron and icosahedron.

The type of a map

In an orientably regular map $\mathcal{M}$, all vertices have the same valency n, and all faces have the same valency (number of sides) m, so

$$x^n = y^2 = z^m = xyz = 1,$$

and the map has type $\{m, n\}$ (notation due to Coxeter).

Thus the tetrahedron has type $\{3, 3\}$, the cube and the octahedron have types $\{4, 3\}$ and $\{3, 4\}$, and the dodecahedron and the icosahedron have types $\{5, 3\}$ and $\{3, 5\}$.

More generally, the orders n and m of x and z are the least common multiples of the valencies of the vertices and faces.

To restrict attention to maps of type $\{m, n\}$, add the relations $X^n = Z^m = 1$ to Δ, replacing $\Delta(\infty, 2, \infty)$ with $\Delta(n, 2, m)$.

We can also take m or $n = \infty$, ignoring the corresponding relation.

Thus for triangular maps, with no restriction on vertex valencies, we can use $\Delta(3, 2, \infty) = \langle X \rangle * \langle Y \rangle \cong C_3 * C_2 \cong PSL_2(\mathbb{Z})$.

Calculating the genus

If $\mathcal{M}$ is an orientably regular finite map of type $\{m, n\}$ with monodromy group G ($\cong A$) the numbers of vertices, edges and faces are

$$V = \frac{|G|}{n}, \quad E = \frac{|G|}{2}, \quad F = \frac{|G|}{m},$$

so the surface has Euler characteristic

$$\chi = V - E + F = |G| \left(\frac{1}{n} - \frac{1}{2} + \frac{1}{m} \right)$$

and genus

$$g = 1 - \frac{\chi}{2} = 1 - \frac{|G|}{2} \left(\frac{1}{2} - \frac{1}{m} - \frac{1}{n} \right).$$

Chirality

An isomorphism $\mathcal{M} \to \mathcal{M}'$ is a bijection $\Phi \to \Phi'$ and isomorphism $G \to G'$ such that $(\alpha x)' = \alpha' x'$ and $(\alpha y)' = \alpha' y'$ for all $\alpha \in \Phi$.

If $\mathcal{M}$ is an oriented map, its mirror image $\overline{\mathcal{M}}$ is the map with the same set Φ of arcs, and x and y inverted (in fact, $y = y^{-1}$).

$\mathcal{M}$ is reflexible if $\mathcal{M} \cong \overline{\mathcal{M}}$ (equivalently, G has an automorphism inverting x and y), otherwise $\mathcal{M}$ and $\overline{\mathcal{M}}$ form a chiral pair.

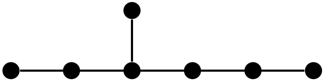
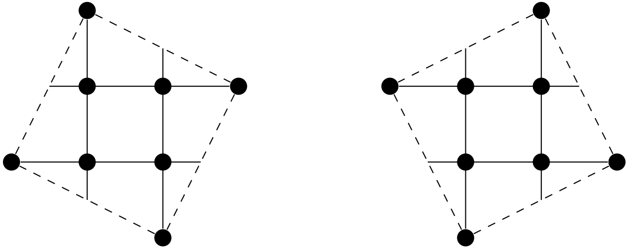


Figure: A chiral pair of maps

An orientably regular map is fully regular (or simply regular) if it is also reflexible. The platonic solids are fully regular, but some maps on the torus, and on surfaces of higher genus, are not.

Example of chirality

In each diagram, identifying opposite sides of the outer square gives an orientably regular map of type $\{4, 4\}$ on the torus (an embedding of the complete graph K_5). They are mirror images, but not isomorphic (see the Exercise), so they form a chiral pair.



Exercise. Show that these two maps are non-isomorphic and orientably regular. Find their automorphism groups.

Fully regular maps

Orientably regular maps $\longleftrightarrow M \triangleleft \Delta = \Delta(\infty, 2, \infty)$.

$\mathcal{M}$ is fully regular iff M is invariant under the automorphism $\iota : X \mapsto X^{-1}, Y \mapsto Y^{-1}$ of Δ.

The extended triangle group $\Delta[\infty, 2, \infty]$ contains Δ with index 2. Its element R_2 induces ι (by conjugation) on the subgroup Δ, so:

Fully regular maps $\longleftrightarrow M \leq \Delta$ such that $M \triangleleft \Delta[\infty, 2, \infty]$.
Such maps $\mathcal{M}$ have full automorphism group $\Delta[\infty, 2, \infty]/M$.

Chiral maps $\longleftrightarrow$ pairs of subgroups $M \leq \Delta$ transposed by ι.

There is a similar but more general theory of maps, based on extended triangle groups, where the surface can be non-orientable, or have non-empty boundary, or both.

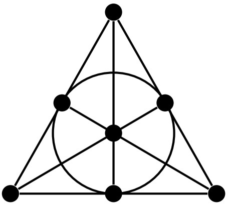
Example. The quotient of a platonic solid by an antipodal automorphism is a regular map on the real projective plane. The quotient by a reflection is a map on the closed disc.

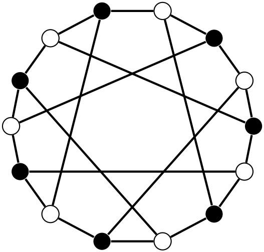
Hypermaps

A hypermap is an embedding of a hypergraph in a surface.

A hypergraph is a set of points (or hypervertices) with a set of non-empty subsets called blocks (or hyperedges). It can be represented by its Levi graph: a bipartite graph with black vertices $\longleftrightarrow$ points, white vertices $\longleftrightarrow$ blocks, edges $\longleftrightarrow$ incidence. Conversely, every bipartite graph represents a hypergraph.

Example. The diagram shows the Fano plane $\mathbb{P}^2(\mathbb{F}_2)$, with blocks $=$ lines, and its Levi graph, the Heawood graph.

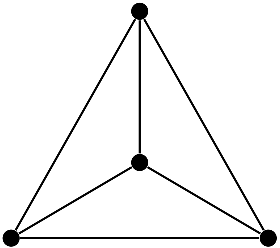


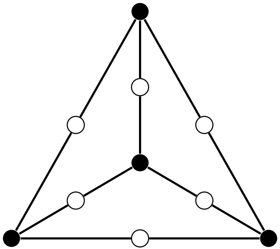


Hypermaps $\longleftrightarrow$ bipartite graphs embedded in surfaces.

Any map $\mathcal{M}$ can be regarded as a hypermap $\mathcal{H}$: colour its vertices black, and put a white vertex on each edge (or at the free end of each semi-edge), so white vertices of $\mathcal{H} \longleftrightarrow$ edges of $\mathcal{M}$.

Example Here is the tetrahedron, as a map and as a hypermap.



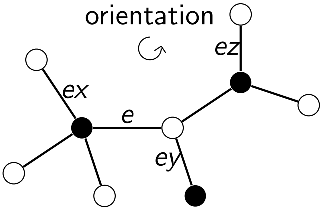


Hypermaps and permutations

We can describe hypermaps by permutations. Since the blocks can have any size, we omit the relation $Y^2 = 1$, and regard an oriented hypermap $\mathcal{H}$ as a permutation representation of the triangle group

$$\Delta = \Delta(\infty, \infty, \infty) = \langle X, Y, Z \mid XYZ = 1 \rangle = \langle X, Y \mid - \rangle \cong F_2.$$

X and Y induce permutations x and y of the edges, following the orientation around their incident black and white vertices; Z induces the permutation $z = (xy)^{-1} = y^{-1}x^{-1}$ rotating edges two steps around their incident faces, so the black and white vertices and faces correspond to the cycles of x, y and z on the edges.



The type of a hypermap

A hypermap has type (l, m, n) if l, m and n are the orders of x, y and z, i.e., the least common multiples of the valencies of the black and white vertices and the faces of the bipartite map.

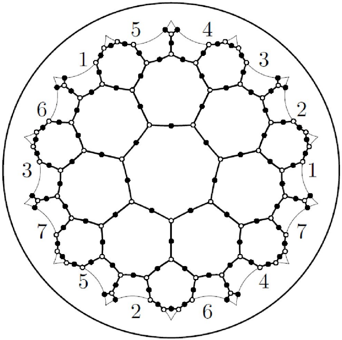
(Every face is a $2k$-gon for some k, corresponding to a cycle of z of length k, so we regard its valency as k.)

Hypermaps of type (l, m, n) correspond to permutation representations of the triangle group

$$\Delta(l, m, n) = \langle X, Y, Z \mid X^l = Y^m = Z^n = XYZ = 1 \rangle.$$

As with maps, we can take any of $l, m, n = \infty$ and ignore the corresponding relation.

Here is a hypermap of type $(2, 3, 7)$, on a surface of genus 3, formed by identifying sides of the 14-gon according to the numbering. We will return to this important example later.



The concepts and results we have considered for maps, concerning automorphisms, regularity, chirality, etc, all carry over to hypermaps in the obvious way. Indeed, maps are simply hypermaps with m dividing 2.