Lecture 5: Dessins d'enfants (G2S2, Novosibirsk, 2018)

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Introduction

Riemann showed that compact Riemann surfaces are essentially the same as complex algebraic curves, defined by polynomials with coeffficients in $\mathbb{C}$. (In modern terms, the categories are equivalent.)

I shall illustrate this for genus 1, where tori are equivalent to elliptic curves.

In the early 1980s, Grothendieck wondered which Riemann surfaces were defined over the subfield $\overline{\mathbb{Q}} \subset \mathbb{C}$ of algebraic numbers.

In fact this question had already been answered a few years earlier by Belyĭ, as a lemma in Galois Theory.

Grothendieck reinterpreted his solution in terms of maps on surfaces, which he called dessins d'enfants, or simply dessins.

I shall outline this connection, using triangle groups, and illustrate it with some examples, including the Fermat curves.

Riemann surfaces and algebraic curves

The following theorem is essentially due to Riemann:

Theorem

A Riemann surface X is compact if and only if it is a (projective) algebraic curve

Here 'projective' means contained in some projective space $\mathbb{P}^n(\mathbb{C})$, though it is sometimes easier to work in affine space $\mathbb{C}^n$, with special arguments for 'points at infinity'; 'algebraic' means defined by polynomial equations, and 'curve' means (in algebraic geometry) 1-dimensional over $\mathbb{C}$, hence 2-dimensional over $\mathbb{R}$.

A compact Riemann surface is homeomorphic to a sphere with g handles for some $g \geq 0$, called its genus.

The only Riemann surface of genus 0 is the Riemann sphere

$$X = \mathbb{P}^1(\mathbb{C}) = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

The first interesting case is genus 1, where each torus is equivalent to an elliptic curve, and vice versa.

From an elliptic curve to a torus

Let $p(z) \in \mathbb{C}[z]$ be a cubic polynomial with distinct roots $a, b, c \in \mathbb{C}$, and let X be the elliptic curve $w^2 = p(z)$, the Riemann surface of the 2-valued function $w = \sqrt{p(z)}$.

The projection $X \to \hat{\mathbb{C}}$, $(z, w) \mapsto z$ realises X as a 2-sheeted covering of the sphere $\hat{\mathbb{C}}$, branched over a, b, c and ∞.

If we cut $\hat{\mathbb{C}}$ along two disjoint paths P_1 and P_2 from a to b and from c to ∞, we can define two single-valued branches of $\sqrt{p(z)}$ on the domain $D := \hat{\mathbb{C}} \setminus (P_1 \cup P_2)$.

Then X is formed by joining two copies of D, one for each branch of $\sqrt{p(z)}$, across the cuts P_1 and P_2.

Each copy of D is topologically a cylinder, so joining them by identifying their boundary components gives a torus, of genus 1.

From a torus to an elliptic curve

Let X be a torus $\mathbb{C}/\Lambda$ for some lattice $\Lambda \subset \mathbb{C}$.

The Weierstrass $\wp$-function for Λ is the meromorphic function

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega}{}' \Big(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \Big),$$

where Σ'_ω denotes summation over all $\omega \in \Lambda \setminus \{0\}$.

One can show that $\wp$ is periodic, i.e. $\wp(z+\omega) = \wp(z)$ for all $z \in \mathbb{C}$ and $\omega \in \Lambda$, so it is an elliptic function with respect to Λ.

Hence $\wp$ induces a meromorphic function $\wp : X = \mathbb{C}/\Lambda \to \hat{\mathbb{C}}$.

Now $\wp$ satisfies a differential equation $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$, where the polynomial $p(z) = 4z^3 - g_2 z - g_3$ has distinct roots.

Points $t + \Lambda \in \mathbb{C}/\Lambda$ correspond to points $(w, z) = (\wp'(t), \wp(t))$ on the elliptic curve $w^2 = p(z)$.

(Compare with parametrizing the circle $\mathbb{R}/2\pi\mathbb{Z}$ by $(\cos t, \sin t)$.)

Genus $g > 1$

The situation for genus $g > 1$ is similar, but more complicated, so I will avoid details.

Instead of the complex plane $\mathbb{C}$ we use the hyperbolic plane $\mathbb{H}$.

The lattice Λ is replaced with a surface group

$$\Gamma = \langle A_i, B_i \ (i = 1, \ldots, g) \mid \prod_{i=1}^{g} [A_i, B_i] = 1 \rangle \leq \mathrm{Aut}\, \mathbb{H} = PSL_2(\mathbb{R}),$$

such that $X \cong \mathbb{H}/\Gamma$.

The elliptic functions $\wp$ and $\wp'$ are replaced with automorphic functions, that is, meromorphic functions on $\mathbb{H}$ invariant under Γ.

A fundamental parallelogram for Λ, with opposite sides identified to form a torus, is replaced with a fundamental region for Γ, for instance a $4g$-gon in $\mathbb{H}$ with sides identified to form X.

Fields of definition

A compact Riemann surface X is defined over a subfield K of $\mathbb{C}$ if it can be defined (as an algebraic curve) by polynomials with coefficients in K. It is an important problem to characterise the compact Riemann surfaces defined over various subfields $K \subset \mathbb{C}$.

The case $K = \mathbb{R}$ is easy: X is defined over $\mathbb{R}$ if and only if it has an anti-conformal (angle-preserving, orientation-reversing) automorphism of order 2, such as complex conjugation.

The case $K = \mathbb{Q}$ is very hard, with only partial results for genus 1.

We will be interested in the case where K is the field $\overline{\mathbb{Q}}$ of algebraic numbers, those complex numbers (such as i and $\sqrt[3]{2}$) which are roots of polynomials in $\mathbb{Q}[x]$, or equivalently in $\mathbb{Z}[x]$.

Belyĭ's Theorem

If X is compact, a non-constant meromorphic function $f : X \to \hat{\mathbb{C}}$ is an n-sheeted branched covering for some degree $n = \deg(f) \geq 1$.

This means that $|f^{-1}(w)| = n$ for all but finitely many $w \in \hat{\mathbb{C}}$.

The exceptional points w, where $1 \leq |f^{-1}(w)| < n$, are the critical values of f, where the covering is branched.

Theorem (Belyĭ, 1979)

A compact Riemann surface X is defined over $\overline{\mathbb{Q}}$ if and only if there is a non-constant meromorphic function $\beta : X \to \hat{\mathbb{C}}$ with at most three critical values.

Then β is called a Belyĭ function, and (X, β) a Belyĭ pair.

In fact, Belyĭ proved only that this condition is necessary, and simply gave a very short sketch of a proof that it is sufficient.

This part of the proof, which is much harder, was completed later by Wolfart (1997) and Koeck (2004).

The critical values

$$\mathrm{Aut}\,\hat{\mathbb{C}} = PSL_2(\mathbb{C}) = \{z \mapsto \frac{az+b}{cz+d} \mid a,b,c,d \in \mathbb{C}, ad-bc=1\}$$

acts 3-transitively on $\hat{\mathbb{C}}$, i.e. transitively on distinct ordered triples, so by composing a Belyĭ function β with an automorphism one can assume that the critical values are in $\{0,1,\infty\}$.

Example If $n \in \mathbb{Z} \setminus \{0\}$ then $\beta_n : z \mapsto z^n$ is a Belyĭ function $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$, with critical values in $\{0,\infty\}$.

Exercise Show that if m and n are positive integers then

$$\beta = \beta_{m,n} : z \mapsto \frac{(m+n)^{m+n} z^m (1-z)^n}{m^m n^n}$$

is a Belyĭ function $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$, and that $\beta(\{0,1,a,\infty\}) \subseteq \{0,1,\infty\}$ where $a = m/(m+n)$.

Belyĭ functions on Fermat curves

The Fermat curve $w^n + z^n = 1$ is a compact Riemann surface X of genus $(n-1)(n-2)/2$, defined over $\mathbb{Q} \subset \overline{\mathbb{Q}}$.

The projection $\pi : (z, w) \mapsto z$ has critical values at the nth roots of 1, so composing π with $\beta_n : z \mapsto z^n$ gives a Belyĭ function

$$\beta = \beta_n \circ \pi : X \to \hat{\mathbb{C}},\ (z, w) \mapsto z^n,$$

of degree n^2, branched over $b = 0, 1, \infty$, where $|\beta^{-1}(b)| = n$.

Belyĭ's proof of the 'necessary' part of his Theorem is similar. Given a Riemann surface X defined over $\overline{\mathbb{Q}}$, first use a coordinate projection onto $\hat{\mathbb{C}}$, with algebraic numbers as critical values. Next use minimal polynomials to send these to rational numbers. Then use Möbius transformations, together with functions like $\beta_{m,n}$ which map four rational numbers to three, to reduce the number of critical values to three.

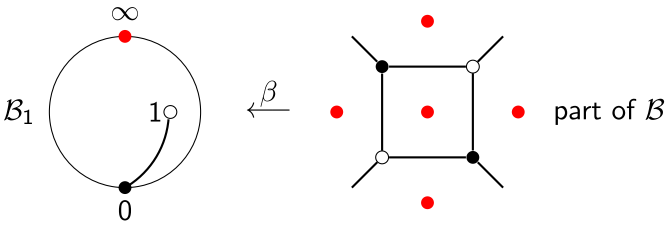
From Belyĭ pairs to dessins

Any Belyĭ pair (X, β) yields a dessin, in the form of a bipartite map $\mathcal{B} = \beta^{-1}(\mathcal{B}_1)$ on X, where $\mathcal{B}_1$ is the trivial bipartite map on $\hat{\mathbb{C}}$.

This has two vertices, at 0 and 1, coloured black and white, joined by an edge along the unit interval $I = [0, 1] \subset \mathbb{R}$ (see below).

Then $\beta^{-1}(I)$ is the embedded graph, and $\beta^{-1}(0)$, $\beta^{-1}(1)$ and $\beta^{-1}(\infty)$ are the sets of black and white vertices and face-centres (the latter shown in red).

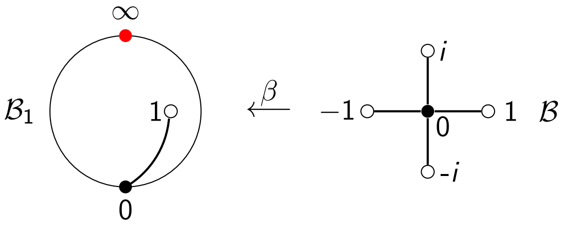
Since β is unbranched over the open interval $(0, 1)$, the single edge I of $\mathcal{B}_1$ lifts to $\deg(\beta)$ edges of $\mathcal{B}$, meeting only at their end-points.



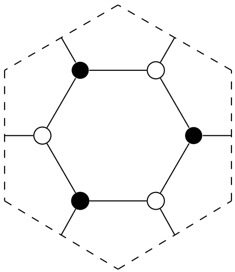
Example Let $\beta = \beta_n : \hat{\mathbb{C}} \to \hat{\mathbb{C}}, z \mapsto z^n$ as earlier, where $n \geq 1$.
Then $\mathcal{B} = \beta^{-1}(\mathcal{B}_1)$ is a bipartite map on $X = \hat{\mathbb{C}}$.

It has a single black vertex at 0, joined by straight-line edges to white vertices at the nth roots of 1, and a single $2n$-gonal face.

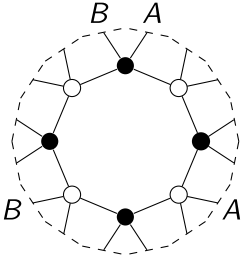
The case $n = 4$ is shown here, with $\mathcal{B}$ drawn in the plane $\mathbb{C}$.



Example The dessin corresponding to the Belyĭ function $(z, w) \mapsto z^n$ on the Fermat curve of degree n is an orientably regular embedding of the complete bipartite graph $K_{n,n}$.



(a) $n = 3$



(b) $n = 4$

In (a), identify opposite sides of the outer hexagon to form a torus.

In (b), identify sides of the outer 16-gon, A with A, B with B, etc (using dihedral symmetry to determine the rest), to give an orientable surface of genus 3.

Belyĭ pairs and triangle groups

Any covering $X \to Y$ of Riemann surfaces corresponds to an inclusion $\Gamma \leq \Delta$ of Fuchsian groups, where $X \cong \mathbb{H}/\Gamma$ and $Y \cong \mathbb{H}/\Delta$. (In a few cases we may need to replace $\mathbb{H}$ with $\mathbb{C}$ or $\hat{\mathbb{C}}$.)

In the case of a Belyĭ function $\beta : X \to \hat{\mathbb{C}}$, we need Δ to be cocompact and have genus 0, with at most three elliptic periods, corresponding to the three critical values. Thus Δ, with signature $(0; m_1, m_2, m_3)$, is a cocompact triangle group $\Delta(m_1, m_2, m_3)$. (The periods m_i depend on the branching over the critical values.)

Also, Γ has finite index $|\Delta : \Gamma| = \deg(\beta)$ in Δ.

Thus Belyĭ's Theorem can be restated as:

Theorem

A Riemann surface X is defined over $\overline{\mathbb{Q}}$ if and only $X \cong \mathbb{H}/\Gamma$ for a subgroup Γ of finite index in a cocompact triangle group.

Belyĭ pairs and hypermaps

We have seen that subgroups of triangle groups correspond to oriented hypermaps. Those of finite index correspond to finite hypermaps.

In the case of a Belyĭ function β, the hypermap corresponding to the inclusion $\Gamma \leq \Delta$ is (when represented as a bipartite map) simply the map $\mathcal{B} = \beta^{-1}(\mathcal{B}_1)$ we have just described.

Belyĭ's Theorem can therefore be restated as:

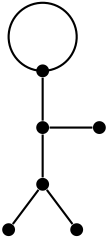
Theorem

A Riemann surface X is defined over $\overline{\mathbb{Q}}$ if and only it is obtained from a finite oriented hypermap.

Thus very complicated structures, such as X, can be represented by very simple combinatorial objects, such as $\mathcal{B}$.

Grothendieck, impressed by their simplicity, called them dessins d'enfants (children's drawings).

Monsieur Mathieu



To show what Grothendieck meant, the portrait shown here is a map $\mathcal{M}$ of genus 0. It automorphism group is trivial, but its monodromy group, a permutation group of degree 12 (the number of directed edges), is the Mathieu group M_{12}, one of the five sporadic simple groups discovered by Mathieu in the mid-19th century. This group has order 95040, and it is the automorphism group of the minimal regular cover $\widetilde{\mathcal{M}}$ of $\mathcal{M}$, a map of type $\{11, 3\}$ and genus 3601 on a curve defined over the field $\mathbb{Q}(\sqrt{-11})$.

The Fermat curves revisited

For any dessin, the elliptic periods m_i of Δ are the least common multiples of the valencies of the black and white vertices and faces.

For the Fermat curves, each valency is n, so $\Delta = \Delta(n, n, n)$.

Γ has index $\deg(\beta) = n^2$ in Δ.

Γ is normal in Δ since $\mathcal{B}$ is orientably regular: its automorphism group Δ/Γ is $C_n \times C_n$, visible in the equation $w^n + z^n = 1$ by multiplying w and z by nth roots of 1.

Exercise Show that $\Delta(n, n, n)$ has a unique normal subgroup with quotient $C_n \times C_n$, namely its commutator subgroup, so $\Gamma = \Delta'$.

For any dessin, $\mathrm{Aut}\,\mathcal{B} \leq \mathrm{Aut}\,X$, and this inclusion could be proper.

Exercise Show that $|\mathrm{Aut}\,X : \mathrm{Aut}\,\mathcal{B}| \geq 6$ for the Fermat curves. (Hint: transform X into the projective curve $x_1^n + x_2^n + x_3^n = 0$.)

In fact, one can show that $|\mathrm{Aut}\,X : \mathrm{Aut}\,\mathcal{B}| = 6$ provided $n \geq 4$.