Lecture 7. Hurwitz Groups and Surfaces (G2S2, Novosibirsk, 2018)

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August 5, 2018

Introduction

In 1890 Schwarz showed that if X is a compact Riemann surface of genus $g > 1$ then its automorphism group is finite.

in 1893 Hurwitz showed that $|\mathrm{Aut}\, X| \leq 84(g-1)$.

The groups attaining this bound are called Hurwitz groups.

I will prove this bound, and look at examples of small genus.

Then I will consider some infinite families of Hurwitz groups.

Finally I will relate these groups to maps and dessins.

Some basic group theory is needed, in particular Sylow's Theorems and the definition and order of the simple groups $PSL_2(q)$. These are summarised in the accompanying notes.

The Hurwitz bound

A compact Riemann surface of genus $g > 1$ has the form $X = \mathbb{H}/K$ for some surface group K of genus g.

It has automorphism group $A \cong N(K)/K$ where $N = N(K)$ is the normaliser of K in $PSL_2(\mathbb{R})$.

K has signature $(g; -)$, so X has area $\mu(\mathbb{H}/K) = 2\pi(2g-2)$.

Being cocompact (since K is), N has signature $(h; m_1, \ldots, m_r)$, so

$$\mu(\mathbb{H}/N) = 2\pi\left(2h - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right)\right).$$

The Riemann–Hurwitz formula $\mu(\mathbb{H}/K) = |N : K|\,\mu(\mathbb{H}/N)$ gives

$$\frac{2g-2}{|A|} = 2h - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right).$$

For fixed g, $|A|$ is maximised iff the right-hand side is minimised.

By elementary case-by-case analysis (exercise!), the RHS of

$$\frac{2g-2}{|A|} = 2h - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right)$$

has least positive value $1/42$, so that $|A| \leq 84(g-1)$, attained iff $h = 0$, $r = 3$ and $m_i = 2, 3$ and 7, or equivalently $N = \Delta(2, 3, 7)$.

Thus we have proved:

Theorem (Hurwitz, 1893)

If X is a compact Riemann surface X of genus $g > 1$, then

$$|\mathrm{Aut}\, X| \leq 84(g-1),$$

attained if and only if $X \cong \mathbb{H}/K$ where K is a proper normal subgroup of finite index in the triangle group $\Delta = \Delta(2, 3, 7)$. In this case $\mathrm{Aut}\, X \cong \Delta/K$.

Characterizing Hurwitz groups

The groups and surfaces attaining the Hurwitz upper bound are called Hurwitz groups and Hurwitz surfaces.

Corollary

A group is a Hurwitz group if and only if it is a non-identity finite quotient of the triangle group $\Delta = \Delta(2,3,7)$.

Hence a non-identity finite group is a Hurwitz group if and only if it has generators x, y and z satisfying

$$x^2 = y^3 = z^7 = xyz = 1.$$

Making x and y commute implies $x = y = z = 1$ (exercise!), so Hurwitz groups are perfect (no non-identity abelian quotients).

The smallest Hurwitz groups are non-abelian finite simple groups, and all others are covering groups of them.

Since the classification of finite simple groups ($\sim$ 1980), many of them have been shown to be (or not to be) Hurwitz groups.

Hurwitz groups of small genus

Theorem

There is no Hurwitz group of genus 2.

Proof. A Hurwitz group A of genus 2 has order $|A| = 84 = 2^2.3.7$.

By Sylow's Theorems its Sylow 7-subgroups have order 7; the number n_7 of them divides $|A|$ with $n_7 \equiv 1 \bmod (7)$, so $n_7 = 1$.

Thus A has a unique (and hence normal) Sylow 7-subgroup S.

As a non-identity finite quotient of Δ, $B := A/S$ is a Hurwitz group, whereas $|B| = 12 \neq 84(g-1)$ for any integer g. $\square$

(In fact, the largest automorphism group of a Riemann surface of genus 2 is $GL_2(3)$, of order 48, arising as a quotient of $\Delta(2,3,8)$. The corresponding surface is a double covering of the Riemann sphere, branched over the six face-centres of a cube.)

Theorem (Klein, 1878)

There is a Hurwitz group of genus 3.

Proof. The simple group $A = PSL_2(7) = SL_2(\mathbb{F}_7)/\{\pm I\}$ has order 168, so it could be a Hurwitz group of genus 3. Its elements

$$x = \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad y = \pm\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad z = \pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

satisfy $x^2 = y^3 = z^7 = xyz = 1$ (exercise!).

The subgroup H they generate is a Hurwitz group of genus $g \leq 3$.

There is no Hurwitz group of genus 2, so $g = 3$.

Hence $|H| = 168$, so $H = A$, that is, x, y and z generate A. $\square$

Klein's quartic curve

One can show that $\Delta(2,3,7)$ has a unique normal subgroup K of index 168, so $PSL_2(7)$ is the only Hurwitz group of genus 3, and $X = \mathbb{H}/K$ is the only Hurwitz surface of that genus.

As a complex algebraic curve it is given by the equation

$$x^3y + y^3z + z^3x = 0$$

in homogeneous coordinates $[x, y, z]$ in the complex projective plane $\mathbb{P}^2(\mathbb{C})$; it is known as Klein's quartic curve.

As a Riemann surface, X can be obtained by compactifying $\mathbb{H}/\Gamma(7)$ (filling in its eight punctures), where $\Gamma(7)$ is the principal congruence subgroup of level 7 in the modular group $\Gamma = PSL_2(\mathbb{Z})$.

Note that $\Gamma/\Gamma(7) \cong PSL_2(7)$, giving an action of $PSL_2(7)$ on X.

X can also be obtained by using the labelling to identify sides of this 14-gon, drawn in the unit disc model of the hyperbolic plane. (This is Klein's original diagram, showing the tessellation by fundamental triangles for $\Delta[2, 3, 7]$.)

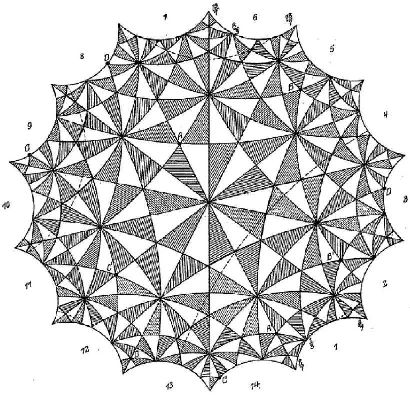


Figure: The unique Hurwitz surface of genus 3

Infinite families of Hurwitz groups

Given one Hurwitz group, one can find infinitely many others.

Let A be a Hurwitz group of genus g, corresponding to a normal subgroup K of index $84(g-1)$ in Δ.

For any $m \geq 2$, let $L = K'K^m$ be the subgroup of K generated by its commutators and mth powers. This is a characteristic subgroup of K, and K is a normal subgroup of Δ, so L is normal in Δ. Now

$$K = \langle A_i, B_i \ (i = 1, \ldots, g) \mid \prod_{i=1}^{g} [A_i, B_i] = 1 \rangle,$$

so by making its generators commute and have order m we see that

$$K/L \cong C_m \times \cdots \times C_m \quad (2g \text{ factors}),$$

a group of order m^{2g}. Thus L has finite index $84(g-1)m^{2g}$ in Δ, so Δ/L is a Hurwitz group of genus $1 + (g-1)m^{2g}$.

This idea (Macbeath, 1961) is known as 'the Macbeath trick'.

Hurwitz groups of type $PSL_2(q)$

Theorem (Macbeath, 1969)

The group $PSL_2(q)$ is a Hurwitz group if and only if

1. $q = 7$, *or*
2. $q = p$ *for some prime* $p \equiv \pm 1 \bmod (7)$, *or*
3. $q = p^3$ *for some prime* $p \equiv \pm 2$ *or* $\pm 3 \bmod (7)$.

In cases (1) and (3) there is one Hurwitz curve for each q, but in case (2) there are three, corresponding to three normal subgroups of Δ with quotient $PSL_2(q)$.

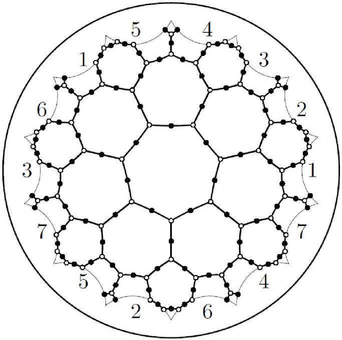
($PSL_2(8)$, of genus 7, was found in 1899 by Burnside and Fricke.)

Dirichlet's theorem on primes in an arithmetic progression states that if a and b are mutually coprime integers, there are infinitely many primes $p \equiv a$ mod (b); thus there are infinitely many primes satisfying each of the congruences in Macbeath's theorem.

Hurwitz groups and maps

Since $\Delta(2,3,7) \cong \Delta(3,2,7)$, Hurwitz groups are automorphism groups of orientably regular finite maps of type $\{7,3\}$.

For example, $PSL_2(7) \cong \mathrm{Aut}\,\mathcal{K}$ where $\mathcal{K}$ is Klein's map, shown here on Klein's quartic curve as a dessin (bipartite map) of type $(2,3,7)$; ignoring the black vertices of valency 2 gives the required trivalent map of 7-gons.



Hurwitz groups and fields of definition

By Belyĭ's Theorem, Hurwitz surfaces are defined over $\overline{\mathbb{Q}}$, being uniformised by subgroups of finite index in a triangle group Δ.

This is obvious for Klein's quartic, which has defining equation

$$x^3y + y^3z + z^3x = 0.$$

Recall Macbeath's result that $PSL_2(p)$ is a Hurwitz group for each prime $p \equiv \pm 1 \bmod (7)$, with three Hurwitz surfaces for each p.

Streit (2000) showed that they are defined over the real cyclotomic field

$$K = \mathbb{R} \cap \mathbb{Q}(\zeta) = \mathbb{Q}(\zeta + \overline{\zeta}) = \mathbb{Q}(2\cos(2\pi/7)), \quad (\zeta := e^{2\pi i/7}),$$

and form an orbit of the absolute Galois group $\mathbb{G}$, acting as the Galois group $G_K \cong A_3 \cong C_3$ of K. These dessins have genus

$$1 + \frac{p(p^2-1)}{168}.$$

Recent results on Hurwitz groups and dessins

Theorem (Conder, 1980)

The alternating group A_n *is a Hurwitz group for all* $n \geq 168$*, and for a known set of values* $n < 168$ *(starting with* $n = 15$*).*

Theorem (Lucchini, Tamburini and J. S. Wilson, 2000)

The group $SL_n(q)$ *(and hence also the simple group* $PSL_n(q)$*) is a Hurwitz group for all prime powers* q *if* $n \geq 287$*.*

Theorem (Conder, R. A. Wilson, Woldar, etc)

Of the 26 sporadic simple groups, twelve are Hurwitz groups, including the Monster.

Theorem (González-Diez and Jaikin-Zapirain, 2015)

The absolute Galois group $\mathbb{G}$ *acts faihfully on Hurwitz dessins (more generally on dessins of any given hyperbolic type).*