# Lecture 8: Graphs from the Point of view of Riemann Surfaces (G2S2, Novosibirsk, 2018)

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## Graphs and Riemann surfaces

The theory of Riemann surfaces was founded in classical works by B. Riemann and A. Hurwitz. We note that originally Riemann surface was defined as a branched covering over the sphere. Over the last decade, a few discrete versions of the theory of Riemann surfaces were created. For these theories, various versions of the Riemann-Hurwitz formula and the Riemann-Roch theorem have been obtained. Many other theorems of the classical theory of Riemann surfaces have also been carried over to the discrete case.

1. Bacher, R., P. de la Harpe, and Nagnibeda,T., 1997
2. H. Urakawa, H, 2000
3. Baker, M., Norine, S., 2009
4. Caporaso, L., 2011
5. Corry, S., 2013

In these theories, the role of Riemann surfaces is played by graphs, while the branched coverings are replaces by harmonic morphisms of graphs.

# Graphs and Riemann surfaces

*Dictionary*

1. Riemann surface $\iff$ Finite connected graph
2. Riemann surface with a border $\iff$ Finite connected graph with semi-edges
3. Holomorphic map (branched covering) $\iff$ Harmonic map (quasi-covering)
4. The sphere $\iff$ Tree
5. Torus (= one "hole" surface) $\iff$ Flower (= one cycle graph)
6. Genus ($\sharp$ of "holes") $\iff$ Genus ($\sharp$ of independent loops)
7. Conformal automorphism $\iff$ Automorphism acting free on arcs

# Basic Definitions

## Definitions and basic properties

In this section we introduce the notion of a graph with multiple edges, loops and semi-edges. That is a slightly more general notion of a graph. This gives us a way to define the action of group on the graph with multiple edges and loops. Also, we are interesting in the group actions with fixed edges, as well as with invertible edges. The factor space of such an action, in general, is not necessary a graph. But, it can be recognised as a graph with semi-edges.

## Basic Definitions

Following (Malnič A., Nedela R., Škoviera M., 2000) we define a *graph with semi-edges* is an ordered quadruple $X = (D, V; I, \lambda)$ where $D = D(X)$ is a set of *darts*, $V = V(X)$ is a nonempty set of *vertices*, which is required to be disjoint from $D$, $I$ is a mapping of $D$ onto $V$, called the *incidence function*, and $\lambda$ is an involutory permutation of $D$, called the *dart-reversing* involution. For convenience or if $\lambda$ is not explicitly specified we sometimes write $\bar{x}$ instead of $\lambda x$. Intuitively, the mapping $I$ assigns to each dart its *initial vertex*, and the permutation $\lambda$ interchanges a dart and its reverse. The *terminal vertex* of a dart $x$ is the initial vertex of $\lambda x$. The 2-orbits of $\lambda$ are called *edges*. The 1-orbits of $\lambda$ are called *semi-edges*. An edge is called a *loop* if $\lambda x \neq x$ and $I\lambda x = Ix$.

## Graphs with semiedges

We identify the set of edges $E(X)$ of $X$ with the following set of unordered pairs of darts:

$$E(X) = \{\{x, \bar{x}\} : x \in D(X),\, x \neq \bar{x}\}.$$

We will refer to the vertices $Ix$ and $I\bar{x}$ as *endpoints* of the edge $\{x, \bar{x}\}$. In a similar way, the set of semi-edges $S(X)$ of $X$ is identified with the set

$$S(X) = \{\{x\} : x \in D(X),\, x = \bar{x}\}.$$

A *directed* edge of $X$ is an ordered pair $(x, \bar{x})$, where $x \in D(X)$ and $x \neq \bar{x}$. We note that all edges $\{x, \bar{x}\} \in E(X)$, including loops, are provided by two directed edges $(x, \bar{x})$ and $(\bar{x}, x)$.

# Graphs with semiedges

The group $\mathrm{Aut}(X)$ of automorphisms of $X$ is a subgroup of $\mathbf{S}_{D(X)}$ leaving invariant each of the sets $V(X)$, $E(X)$, $S(X)$ and preserving incidence. We say that a group $G$ *acts* on $X$ if $G$ is a subgroup of $\mathrm{Aut}(X)$.

Let $X$ be a finite connected graph. We define the (homological) *genus* of $X$ to be the number

$$g(X) = 1 - |V(X)| + |E(X)|.$$

Recall that $g(X)$ coincides with the Betti number of $X$ that is the rank of the first homology group $\mathrm{H}_1(X, \mathbb{Z})$. Let $G$ be a finite group acting on the graph $X$. An edge $\{x, \bar{x}\} \in E(X)$ is said to be *invertible* (or *reversible*) by $G$ if there is an element $g \in G$ such that $g$ sends $x$ to $\bar{x}$ and $\bar{x}$ to $x$. An edge $\{x, \bar{x}\} \in E(X)$ is said to be *fixed* by $G$ if there is a non-trivial element $g \in G$ that fixes $x$ and $\bar{x}$. We say that $G$ acts on $X$ *without edge reversing* if $X$ has no edges invertible by $G$. Also, $G$ acts on $X$ *without fixed edges* if $X$ has no edges fixed by $G$.

## Graphs with semiedges

From now on all graphs are supposed to be finite and connected. However, multiple edges, loops and even semiedges are allowed. We are going to introduce the notion of a harmonic morphism, or, equivalently, the notion of branched graph covering.

Let $\varphi : X \to Y$ be morphism of graphs. Denote by $\mathrm{St}_X(v)$ the *star* formed by vertex $v$ and all darts of $X$ incedent to $v$. We say that $\varphi$ is *uniform at a vertex* $v$ of $X$ if there exists a non-negative integer $m_v$ such that for every dart $y \in \mathrm{St}_Y(\varphi(v)) \setminus S(Y)$, we have $|\varphi^{-1}(y) \cap \mathrm{St}_X(v)| = m_v$. A morphism $\varphi : X \to Y$ is said to be *locally uniform* if it is uniform at each vertex of $X$. For a given locally uniform morphism defined on $X$, the number $m_v$ attached to the vertex $v$ of $X$ will be called the *multiplicity* at $v$.

*A surjective locally uniform morphism $\varphi$ is called to be harmonic.*

# Harmonic morphisms

A group $G$ acts harmonically if $G$ acts fixed point free on the set of darts $D(X)$ of a graph $X$.

Scott Corry and Roman Nedela made the following useful observation

*If a group $G$ acts harmonically on a graph $X$ then the canonical projection $X \to X/G$ is a harmonic mophism.*

That gives us a lot of non-trivial examples of harmonic morphisms.

## Harmonic morphisms

The following lemma plays a crucial role in the theory of harmonic morphisms.

**Lemma**

*Let $\varphi : X \to Y$ be a harmonic morphism. Then there exists a constant $d = deg(\varphi)$ such that for every ordinary dart $x \in D(Y) \setminus S(Y)$ we have $|\varphi^{-1}(x)| = d$.*
*In particular, each edge of $Y$ has exactly $d$ preimages under $\varphi$.*
*Furthermore, $d = \sum_{v \in \varphi^{-1}(w)} m_v$ for any vertex $v$ of $Y$.*

The number $deg(\varphi)$ from the above Lemma will be called the *degree* of a locally uniform morphism $\varphi$. It follows that the degree determines the size of the fibre $\varphi^{-1}(e)$ over each edge $e = \{x, \bar{x}\} \in E(Y)$.

# Harmonic morphisms

A locally uniform morphism of degree 0 will be called *trivial*. Trivial locally uniform morphisms $\varphi : X \to Y$ are characterised by the following properties: $Y$ has just one vertex, and every dart of $X$ is sent into a semi-edge. Note that trivial harmonic morphisms are exactly the graph epimorphisms onto stars.

# Riemann-Hurwitz formula for harmonic morphisms

Define the multiplicity of edges of $X$ in a harmonic morphism $X \to Y$ by setting $m_e = 2$, if $e$ is mapped onto a semi-edge, and setting $m_e = 1$ otherwise.

Theorem (A. Mednykh and R. Nedela, 2015)

*(Riemann-Hurwitz formula) Let $\varphi : X \to Y$ be a harmonic morphism, and let $g = g(X)$ and $\gamma = g(Y)$ are the respective genera of $X$ and $Y$. Then*

$$g - 1 = \deg(\varphi)(\gamma - 1) + \sum_{v \in V}(m_v - 1) + \sum_{e \in E}(m_e - 1),$$

*where $V, E$ is the set of vertices and the set of edges of $X$ respectively.*

# Riemann-Hurwitz formula for graph coverings

The following statement immediately follows from the Riemann-Hurwitz formula.

**Theorem (Schreier formula)**

*Let $\varphi : G \to G'$ be a graph covering. Suppose that $G$ and $G'$ are graphs of genera $g$ and $g'$ respectively. Then we have*

$$g - 1 = deg(\varphi)(g' - 1).$$

## Groups acting on a graph without edge reversing

Our next result is the following theorem for a group $G$ acting on a graph $X$, possibly with fixed edges, but without edge reversing.

**Theorem (M., 2015)**

*Let $X$ be a graph of genus $g$ and $G$ is a finite group acting on $X$ without edge reversing. Denote by $g(X/G)$ genus of the factor graph $X/G$. Then*

$$g - 1 = |G|(g(X/G) - 1) + \sum_{v \in V(X)} (|G^v| - 1) - \sum_{e \in E(X)} (|G^e| - 1),$$

*where $V(X)$ is the set of vertices, $E(X)$ is the set of edges of $X$, $G^x$ stands for the stabiliser of $x \in V(X) \cup E(X)$ in $G$ and $|G^x|$ is the order of a stabiliser.*

## Groups acting on a graph with edge reversing

Our next result is the following theorem.

### Theorem (M., 2015)

*Let $X$ be a graph of genus $g$ and $G$ is a finite group acting on $X$, possibly with edge reversing. Denote by $g(X/G)$ genus of the factor graph $X/G$. Then*

$$g-1=|G|(g(X/G)-1)+\sum_{v\in V(X)}(|G^v|-1)$$

$$-\sum_{e\in E(X)}(|G^e|-1)+\sum_{e\in E^{inv}(X)}|G^e|,$$

*where $V(X)$ is the set of vertices, $E(X)$ is the set of edges of $X$, $G^x$ is the stabiliser of $x\in V(X)\cup E(X)$ in $G$, and $E^{inv}(X)$ is the set of invertible edges of $X$.*

# Hurwitz type theorems

Recall some classical results for Riemann surface theory. For each $g \geq 2$ define

$$N(g) := \max\{|\mathrm{Aut}(S_g)| : \ S_g \text{ is a compact Riemann surface of genus } g\}.$$

Then

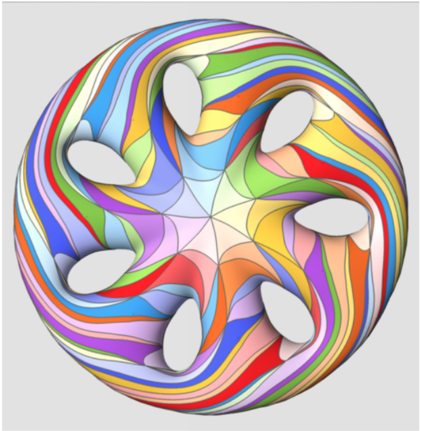
$$8(g+1) \leq N(g) \leq 84(g-1),$$

and these bounds are sharp in the sense that both the upper and lower bound are attained for infinitely many values of $g$. The upper bound was found by Hurwitz (1893). The lower bound was independently obtained by R. Accola (1968) and C. Maclachlan (1969). The curves with the lower bound attached were completely described by R.Accola, C. Maclaclan and R. Kulcarni.

# Hurwitz group and Hurwitz Surface

The Riemann surface with $|\mathrm{Aut}(S_g)| = 84(g-1)$ is called a *Hurwitz surface.* The respective group is a *Hurwitz group.* It was shown by F. Klein ($< 1900$) that the quartic curve, $x^3y + y^3z + z^3x = 0$ of genus 3, admits the group $\mathrm{PSL}_2(7)$ of order $168 = 84(3-1)$ as its full group of conformal automorphisms. It is characterised as the curve of smallest genus realising the upper bound $84(g-1)$ for genus $g > 1$. The next example of Hurwitz curve of genus 7 was done by A. M. Macbeath (1961). It has a beautiful picture given below.

# Hurwitz surface



# Harmonic group action on graphs

*Let $X$ be a graph without semi-edges. A group $G$ act on $X$ harmonically if it acts semi-regular on arcs of $X$.*

Denote by $M(g)$ maximum size of a harmonic group action on any graph of genus $g \geq 2$.

Theorem (Scott Corry, 2011)

*For $g \geq 2$ we have*

$$4(g-1) \leq N(g) \leq 6(g-1).$$

*The upper and lower bound are attained for infinitely many values of $g$.*

Recent paper by Scott Corry (2013) states that maximal graph groups $G$ with $|G| = 6(g-1)$ are exactly the finite quotients of the modular group $\Delta(2,3,\infty) =< x, y | x^2 = y^3 = 1 >$ of size at least 6.

# Oikawa theorem

In 1956 Kotaro Oikawa proved the following theorem.

**Theorem (Oikawa, 1956)**

*Let $S_g$ be a closed Riemann surface of genus $g$ and $A$ is a finite subset of $S_g$ consisting of $|A| \geq 1$ elements. Suppose that $2g - 2 + |A| > 0$ and $G$ is a group of conformal automorphisms of $S_g$ leaving the set $A$ invariant. Then*

$$|G| \leq 12(g-1) + 6|A|.$$

In the next section we find a discrete version of the Oikawa's. Again, the key point of the proof is the Riemann-Hurwitz relation.

## Oikawa's theorem for graphs

Our result for graphs is the following theorem.

**Theorem 4 (R. Nedela, A. Mednykh, 2015)**

*Let $X$ be a graph of genus $g$ and $A$ is a subset of vertices of $X$ consisting of $|A| \geq 1$ elements. Suppose that $g - 1 + |A| > 0$ and $G$ is a finite group acting on $X$ harmonically and leaving the set $A$ invariant. Then*

$$|G| \leq 2(g-1) + 2|A|.$$

The upper bound is sharp and is attained for arbitrary large values of $g$ and $|A|$. So, at least infinitely many often.

# Proof of Oikawa's theorem

Let $G$ be a finite group acting pure harmonically on a graph $X$. For every $\tilde{v} \in V(X)$ denote by $G_{\tilde{v}}$ the stabiliser of $\tilde{v}$ in the group $G$ and by $|G_{\tilde{v}}|$ the order of the stabiliser. Then to each vertex $v \in V(X/G)$ we prescribe the number $m_v = |G_{\tilde{v}}|$, where $\tilde{v} \in \varphi^{-1}(v)$. Since $G$ acts transitively on each fibre of $\varphi$ the numbers $m_v$ are defined correctly.

The following version of the Riemann-Hurwitz formula was given by Baker-Norine (2009) and M. (2015).

*Let $G$ be a finite group acting harmonically on a graph $X$ of genus $g$. Denote by $\gamma$ genus of the factor graph $X/G$. Then*

$$g - 1 = |G|(\gamma - 1 + \sum_{v \in V(X/G)} (1 - \frac{1}{m_v})),$$

*where the numbers $m_v$ are the same as above.*

## Proof of Oikawa's theorem

Preliminary, we establish the following result ("Anti-Hurwitz" Lemma).

### Lemma

*Let $G$ be a finite group acting harmonically on a graph $X$ of genus $g$. Denote by $I \subset V(X)$ a $G$-invariant subset of vertices of $X$. Set $s = |I|$ and $p = |I/G|$. Then*

$$s = |G|(p - \sum_{v \in V(I/G)} (1 - \frac{1}{m_v})),$$

*where the numbers $m_v$ are the same as above.*

# Proof of Oikawa's theorem

**Proof.** Since $p = \sum_{v \in V(I/G)} 1$ we have

$$\begin{aligned} |G|(p - \sum_{v \in V(I/G)} (1 - \frac{1}{m_v})) &= |G|(\sum_{v \in V(I/G)} 1 - \sum_{v \in V(I/G)} (1 - \frac{1}{m_v})) \\ &= |G| \sum_{v \in V(I/G)} \frac{1}{m_v} = \sum_{v \in V(I/G)} \frac{|G|}{m_v} \\ &= \sum_{v \in V(I/G)} |\varphi^{-1}(v)| = |I| = s. \qquad (1) \end{aligned}$$

## Proof of Oikawa's theorem

Now we are able to prove the following proposition.

*Let $G$ be a finite group acting pure harmonically on a graph $X$ of genus $g$. Suppose that $I \subset V(X)$ a $G$-invariant subset of vertices of $X$ and set $s = |I|$ and $p = |I/G|$. Denote by $\gamma$ genus of the factor graph $X/G$. Then*

$$g - 1 + s = |G|(\gamma - 1 + \sum_{v \in V(X/G) - I/G} (1 - \frac{1}{m_v}) + p),$$

*where the numbers $m_v$ are the same as above.*

**Proof.** The desired result is the sum of two formulas. The first one is given by the previous proposition and the second by the Anti-Hurwitz Lemma.

# Proof of Oikawa's theorem

**Proof of of Oikawa's theorem.** We will use the same notations $s = |I|$, $p = |I/G|$ and $m_v$, $v \in V(X/G)$ as above. Since we are going to apply Proposition we are interested only in $v \in V(X/G) - I/G$ with $m_v > 1$. Suppose that there are exactly $r \geq 0$ such vertices. Namely, $v_1, v_2, \ldots, v_r$. We set $m_i = m_{v_i}$, $i = 1, \ldots, r$. Abusing the language we will refer to $v_i$ as branched points of order $m_i$. Then by the above Proposition we have

$$g - 1 + s = |G|(\gamma - 1 + \sum_{i=1}^{r}(1 - \frac{1}{m_i}) + p). \tag{2}$$

## Proof of Oikawa's theorem

We set

$$A = \gamma - 1 + \sum_{i=1}^{r}(1 - \frac{1}{m_i}) + p.$$

Then $|G| = \frac{g-1+s}{A}$. By the assumption of theorem we get $A > 0$. Now, to find the upper bound for $|G|$ we have to find $\min A$ under condition $A > 0$. Now we consider two cases.

1° $\gamma \geq 1$, $p \geq 1$. Then $A \geq 0 + 0 + p \geq 1$ and the minimum value $A = 1$ is attained for $\gamma = 1$, $p = 1$, $r = 0$. In this case $X$ is a regular $G$-covering of a graph of genus $\gamma = 1$ branched over one point.

2° $\gamma = 0$, $p \geq 1$. Then $A \geq \sum_{i=1}^{r}(1 - \frac{1}{m_i}) \geq \frac{1}{2}$. The minimum $A = \frac{1}{2}$ is attained for $\gamma = 0$, $p = 1$, $r = 1$ and $m_1 = 2$. In this case $X$ is a regular $G$-covering of a tree branched over $p + r = 2$ points with branch orders 2 and $s$.

# Proof of Oikawa's theorem

Finally, $A \geq \frac{1}{2}$. As a result we obtain

$$|G| = \frac{g-1+s}{A} \leq 2(g-1+s) = 2(g-1) + 2s.$$

## Two Arakawa's theorems

Now our aim is to find discrete versions of two Arakawa's theorems (2000).

The first one states that if $G$ be a finite group of automorphisms of a compact Riemann surface $X$ of genus $g \geq 2$ and $A$ and $B$ are two disjoint $G$-invariant subsets of $X$ of the orders $|A| \geq |B| \geq 1$ then $|G| \leq 8(g-1) + |A| + 4|B|$.

The second theorem asserts that if $A, B$ and $C$ are three disjoint the $G$-invariant subsets of $X$ with $|A| \geq |B| \geq |C| \geq 1$ then $|G| \leq 2(g-1) + |A| + |B| + |C|$.

# Two Arakawa's theorems

We present a discrete version of the first Arakawa's theorem by the following theorem.

**Theorem 5 (R. Nedela, A. Mednykh and I. Mednykh 2015)**

*Let $X$ be a graph of genus $g \geq 2$ and $A$ and $B$ are two disjoint subsets of vertices of $X$ of the orders $|A| \geq |B| \geq 1$. Suppose that $G$ is a finite group acting harmonically on $X$ and leaving the sets $A$ and $B$ invariant. Then*

$$|G| \leq \frac{3(g-1) + |A| + 3|B|}{2}.$$

Again, the upper bound is sharp and is attained for arbitrary large values of $|A|, |B|$ and $g$.

# Two Arakawa's theorems

A discrete version of the second Arakawa's theorem is given by the following theorem.

**Theorem 6 (R. Nedela, A. Mednykh and I. Mednykh, 2015)**

*Let $X$ be a graph of genus $g \geq 2$ and $A, B$ and $C$ are three disjoint subsets of vertices of $X$ of the orders $|A| \geq |B| \geq |C| \geq 1$. Suppose that $G$ is a finite group acting harmonically on $X$ and leaving the sets $A, B$ and $C$ invariant. Then*

$$|G| \leq \frac{g - 1 + |A| + |B| + |C|}{2}.$$

As in the two previous theorems, the upper bound is sharp and is attained for arbitrary large values of $|A|, |B|, |C|$ and $s$.

## Proof of the second Arakawa's theorems

The proof of this theorem is based on the following considerations. We set $I = A \cup B \cup C, s = |I| = |A| + |B| + |C|$, and $p = |I/G|$. Since $A, B$, and $C$ are pairwise disjoint, it follows that $p \geq 3$. The generalized Riemann-Hurwitz formula yields

$$g-1+s = |G|(\gamma-1+\sum_{i=1}^{r}(1-\frac{1}{m_i})+p) \geq |G|(0-1+0+3) = 2|G|.$$

Since $s = |A| + |B| + |C|$, the latter implies

$$g - 1 + |A| + |B| + |C| \geq 2|G|.$$

# Wiman's theorem

Around the same time, A. Wiman (1895) characterised the curve $S_g : w^2 = z^{2g+1} - 1$ as the unique curves of genus $g$ admitting cyclic automorphism groups of the largest possible order $N = 4g + 2$. The modern proof of this and similar results is contained in the paper by K. Nakagawa (1984).
The Wiman curve $S_g$ can be describe as follows. The regular covering $S_g \to \Sigma = S_g/\mathbb{Z}_N$ is represented by Belyi function $\beta : (w, z) \to z^2$.
Equivalently, one can show that $S_g = \mathbb{H}/\Gamma$ and $\Sigma = \mathbb{H}/\Delta$, where $\Delta = (2, 2g + 1, 2(2g + 1))$ is the triangle group and and $\Gamma \lhd_N \Delta$.

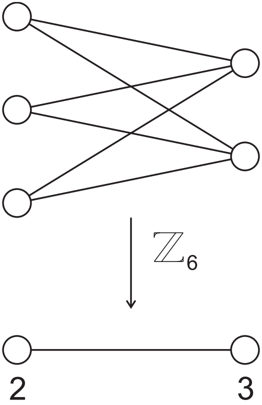
# Wiman's theorem

The aim of the present section is to find a discrete version of the Wiman theorem.

**Theorem 7 (A. Mednykh and I. Mednykh, 2015)**

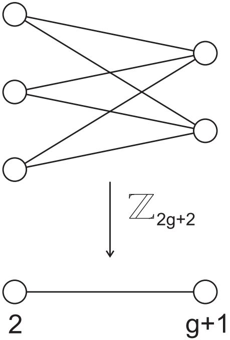
*Let $X$ be a graph of genus $g \geq 2$ and $\mathbb{Z}_N$ is a cyclic group acting harmonically on $X$. Then $N \leq 2g+2$. The upper bound $N = 2g+2$ is attained for any even $g$. In this case, the signature of orbifold $X/\mathbb{Z}_N$ is $(0; 2, g+1)$, that is, $X/\mathbb{Z}_N$ is a tree with two branch points of order $2$ and $g+1$, respectively.*

# Wiman's theorem




$K_{2,3}$ - genus 2 graph

# Wiman's theorem




$K_{2,g+1}$ - genus g graph

g - even