Around symmetries of vertex operator algebras

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§1. A vertex operator algebra (VOA), introduced by Borcherds in 1986 and formulated by Frenkel–Lepowsky–Meurman in 1988, is a vector space V equipped with a countably many binary operations and elements $\mathbf{1}$ and ω subject to a set of axioms. In particular, the element ω generates an action of the Virasoro algebra, which induces a grading on V. If $\dim V_0 = 1$ and $V_1 = 0$, then $B = V_2$ carries a structure of a commutative nonassociative algebra with an invariant symmetric bilinear form:

$$V = \mathbf{C1} \oplus 0 \oplus B \oplus V_3 \oplus \cdots, \quad \cdot : B \times B \longrightarrow B, \quad (\,|\,) : B \times B \longrightarrow \mathbf{C}.$$

The one arising from the Moonshine Module $V^\natural$ agrees with the algebra considered by J. Conway in 1985, a variant of the one constructed by R. Griess in 1981 in proving the existence of the Monster, the largest sporadic finite simple group, and the full automorphism group of $V^\natural$ is indeed the Monster. The algebra B of a VOA V with $\dim V_0 = 1$ and $V_1 = 0$ is called the *Griess algebra* of V.

§2. For a VOA V, let V_ω denote the subVOA generated by the element ω. Assume that the following condition is satified for some n:

$$V^{\mathrm{Aut}\, V} \cap V_\Delta = V_\omega \cap V_\Delta \quad \text{for } \Delta = 0, 1, \ldots, n. \tag{1}$$

When $\dim V_0 = 0$, $V_1 = 0$, and $n = 2m \leq 10$, the traces $\mathrm{Tr}\, R_{a_1} \cdots R_{a_m}$ for the adjoint actions of $a_1, \ldots, a_m \in B$ are expressed by the product and the bilinear form (and a totally antisymmetric quintic form for $m = 5$). For $V = V^\natural$, the condition (1) is satisfied for $n = 11$, and the resulting formulae agree with those previously discovered by S. Norton in 1996 in a different way. As the condition (1) for large n imposes severe constraints on the shape of V, such VOAs are seen to be exceptional.

The formulae are proved by using Casimir type elements κ_Δ or the projection $\pi : V \to V_\omega$, for the assumption (1) implies the properties $\kappa_\Delta \in V_\omega$ and $\mathrm{Tr}|_{V_\Delta}\, o(a) = \mathrm{Tr}|_{V_\Delta}\, \pi(o(a))$ for $\Delta \leq n$, where $o(a)$ is the action of $a \in V$ that preserves the degree. A VOA satisfying the former property is called an *exceptional VOA* by M. Tuite and the latter a *conformal design* by G. Höhn in a broader context in 2008.

§3. In 1996, M. Miyamoto investigated VOAs with the Griess algebra spanned by vectors e that generate actions of the Virasoro algebra with the central charge $c = 1/2$ of certain type, for which the lowest conformal weights are 0 and $1/2$, and found that they induce an action of a 3-transposition group, a group generated by a normal subset D of involutions with $|d_i d_j| \leq 3$ for all $d_i, d_j \in D$. The key observation is that the product $e_p \cdot e_q$ of such vectors is either $2e_q$, 0 or a constant multiple of $e_p + e_q - e_{p\circ q}$ in such a VOA, where $e_{p\circ q}$ is determined by e_p and e_q. Now the algebra is seen to be associated with a partial triple system as in the table below (with $\delta = \gamma = 1/2$ in Miyamoto's setting), and the one associated with a Fischer space is related to the corresponding 3-transposition group.

—	$e_p \cdot e_q$	$(e_p\,\vert\,e_q)$
$p = q$	$2e_p$	$\gamma/2$
$p \perp q$	0	0
$p \sim q$	$\delta(e_p + e_q - e_{p\circ q})/2$	$\delta\gamma/8$

By looking at the least eigenvalue of the colinearlity graph of the Fischer space, the 3-transposition groups that act on a VOA as Miyamoto described are classified.

Miyamoto actually studied the cases such as $V^\natural$ as well where B is spanned by vectors e with $c = 1/2$ whose lowest conformal weights are $0, 1/2$ and $1/16$. The VOAs generated by two such vectors e_1, e_2 are classified by S. Sakuma in 2007. A.A. Ivanov then axiomatized such algebras in 2009 and called them the *Majorana algebras*. In 2015, J.I. Hall, F. Rehren, and S. Shpectorov introduced and studied a class of algebras called the *axial algebras* as a common generalization of the algebras considered so far here.