Compact n-manifolds via $(n+1)$-colored graphs

Michele Mulazzani
University of Bologna, Bologna, Italy
michele.mulazzani(at)unibo.it

This is a joint work with Luigi Grasselli (University of Modena and Reggio Emilia, Italy)

In this work we extend the representation of compact 3-manifolds by 4-colored graphs given in [1], and developed in [2] and [3], to the general dimension $n \geq 3$. In this way, any compact PL n-manifold with (possibly empty) boundary without spherical components can be represented via $(n+1)$-colored graphs (i.e. regular $(n+1)$-valent graphs endowed by a proper edge-coloration with $n+1$ colors). This type of representation was introduced by Pezzana's school in Modena in the seventies, but only for the closed case.

Any $(n+1)$-colored graph Γ induces an n-dimensional quasi-manifold $\widehat{M_\Gamma}$ with singular set S_Γ of dimension $\leq n-3$. By removing from $\widehat{M_\Gamma}$ the interior of the regular neighborhood of S_Γ, we obtain an n-dimensional manifold M_Γ with a boundary without spherical components. Using this construction, any PL compact n-manifold with non-empty non-spherical boundary can be (non-uniquely) represented by $(n+1)$-colored graphs, as in the closed case.

In this context we prove the following result.

Theorem 1. *Let Γ be an $(n+1)$-colored graph and let $\Sigma_c(\Gamma)$ be the $(n+2)$-colored graph obtained from Γ by doubling all edges of a fixed color c, then $\widehat{M}_{\Sigma_c(\Gamma)} = \Sigma(\widehat{M_\Gamma})$, where $\Sigma(\cdot)$ denotes the suspension of the space $\cdot$. Moreover, if M_Γ is not a sphere, then $M_{\Sigma_c(\Gamma)} = M_\Gamma \times [0,1]$.*

As a consequence, we obtain a graph representation Γ_m of the $(m+n)$-manifold $M^n \times B^m$ (where B^m is the m-ball), for any $m \geq 1$, starting from a graph representation Γ of an n-manifold $M^n \neq S^n$, just by adding m parallel edges for any edge of a fixed color of Γ. It is worth noting that the graphs Γ_m and Γ have the same order.

Dipole moves connecting different graphs representing the same manifold, graph techniques for the computation of the fundamental groups of the represented spaces and (boundary-) connected sums of graphs inducing (boundary-) connected sums of the represented manifolds are introduced and/or discussed.

All these constructions and results are included in [4].

References

[1] P. Cristofori, M. Mulazzani, Compact 3-manifolds via 4-colored graphs. *RACSAM* **110** (2016) 395–416.

[2] P. Cristofori, E. Fominykh, M. Mulazzani, V. Tarkaev, 4-colored graphs and knot/link complements. *Results Math.* **72** (2017) 471–490.

[3] P. Cristofori, E. Fominykh, M. Mulazzani, V. Tarkaev, Minimal 4-colored graphs representing an infinite family of hyperbolic 3-manifolds. *RACSAM* (2018) to appear. arXiv:1706.02143

[4] L. Grasselli, M. Mulazzani, Compact n-manifolds via $(n+1)$-colored graphs. Preprint, 2018.