Coherent configurations: linear representation and structure theory

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Content

- Coherent (cellular) algebras and coherent configurations.
- Association scheme, Schur rings
- Ordinary representation theory of coherent algebras.
- Schemes of prime order. Hanaki-Uno Theorem.
- Structure theory of association schemes, closed subsets and factor-schemes

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Binary relations

Let $R, S \subseteq \Omega^2$ be binary relations. Then

- $S^* := \{(\alpha, \beta) \,|\, (\beta, \alpha) \in S\}$;
- S is symmetric (antisymmetric) if $S = S^*$ ($S \cap S^* = \emptyset$ resp.);
- $S(\alpha) := \{\beta \,|\, (\alpha, \beta) \in S\}, (\alpha)S := S^*(\alpha)$;
- $pr_1(S) := \{\alpha \in \Omega \,|\, S(\alpha) \neq \emptyset\}, pr_2(S) := pr_1(S^*)$;
- $RS = \{(\alpha, \beta) \,|\, R(\alpha) \cap S^*(\beta) \neq \emptyset\}$;
- $R^+ = \bigcup_{i=1}^{\infty} R^i$ is the transitive closure of R;
- $1_\Omega := \{(\omega, \omega) \,|\, \omega \in \Omega\}$

Each permutation $g \in \mathrm{Sym}(\Omega)$ is considered as a binary relation. Thus $g(\alpha) = \{\alpha^g\}$ and $g^* = g^{-1}$.

Partitions.

- $\mathcal{P} \vdash \Omega$ means that $\mathcal{P}$ is a partition of Ω;
- $\vec{\mathcal{P}} \vdash \Omega$ means that $\vec{\mathcal{P}}$ is an ordered partition of Ω;
- $\mathcal{P} \sqsubseteq \mathcal{C} \iff \mathcal{C}$ is a refinement of $\mathcal{P}$;
- Lattice operations are denoted as $\mathcal{P} \vee \mathcal{C}$ and $\mathcal{P} \wedge \mathcal{C}$;
- if $\mathcal{P} \vdash \Omega$ then $\mathcal{P}^{\cup}$ denotes the set of all possible unions of elements in $\mathcal{P}$;
- $\mathcal{C} \vdash \Omega^2 \implies \mathcal{C}^* := \{C^* \mid C \in \mathcal{C}\}$;

Coherent configurations (D. Higman, 1970).

Rainbow

Let Ω be a finite set. A pair $\mathcal{X} = (\Omega, \mathcal{C})$ consisting of a finite set Ω and a partition $\mathcal{C} \vdash \Omega \times \Omega$ is called a rainbow if it satisfies the following conditions:

1. $\mathcal{C}(\alpha, \alpha) = \mathcal{C}(\beta, \gamma) \implies \beta = \gamma$;
2. $\mathcal{C}(\alpha, \beta) = \mathcal{C}(\gamma, \delta) \implies \mathcal{C}(\beta, \alpha) = \mathcal{C}(\delta, \gamma)$

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A rainbow $\mathcal{X}$ is called a coherent configuration (briefly c.c.) iff for all $R, S, T \in \mathcal{C}$ the intersection number

$$p_{RS}^{T} = |R(\alpha) \cap S^*(\beta)|$$

does not depend on the choice of $(\alpha, \beta) \in T$.

The numbers $|\Omega|$ and $|\mathcal{C}|$ are called the degree/order and rank of $\mathcal{X}$.

Coherent configurations: a concrete example.

The basic relations and the relations of $\mathcal{X}$ are the relations of $\mathcal{C}$ and of $\mathcal{C}^{\cup}$, resp. The graphs corresponding to the basic relations are called the basic graphs of the configuration.

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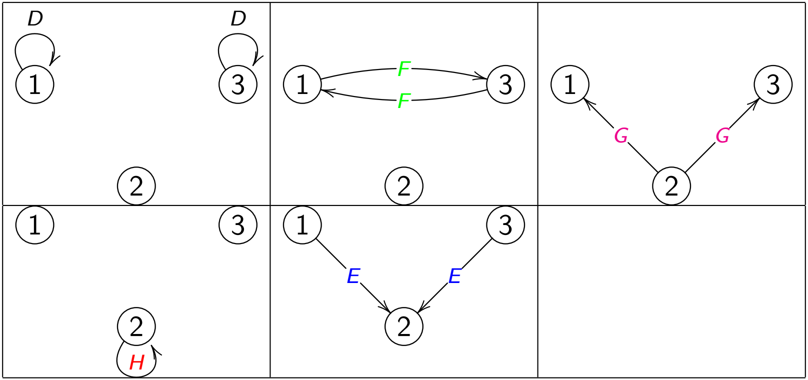

Coherent configurations. Fibers and relations.

A fiber of $\mathcal{X}$ is a set $\Delta \subset \Omega$ such that $1_\Delta \in \mathcal{C}$; the set of all fibers is denoted by $\Phi = \Phi(\mathcal{X})$.
The configuration $\mathcal{X}$ is homogeneous (or an association scheme, or a scheme) if Ω is a unique fiber of $\mathcal{C}$ (equivalently, $1_\Omega \in \mathcal{C}$).

Proposition

Let $\mathcal{X} = (\Omega, \mathcal{C})$ be a co.co. Then

- the set $\mathcal{C}^\cup$ is closed w.r.t. boolean operations;
- $1_\Omega, \Omega^2 \in \mathcal{C}^\cup$;
- $(\mathcal{C}^\cup)^* = \mathcal{C}^\cup$;
- $\mathcal{C}^\cup$ is closed w.r.t. relational product;

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- for any $S \in \mathcal{C}$ and $\alpha \in pr_1(S)$ we have $|S(\alpha)| = p_{SS^*}^T$ where $T = 1_{pr_1(S)}$.
- for any fiber $\Delta \in \Phi$ the set of relations $\mathcal{C}_\Delta := \{C \in \mathcal{C} \mid pr_1(C) = \Delta, pr_2(C) = \Delta\}$ form a homogeneous co.co. on Δ, called a homogeneous constituent of $\mathcal{C}$.

The number $k_S = p_{SS^*}^T$ is called the valency of S.

Exercises.

Proposition

Prove that if the symmetrization $\{C \cup C^* \mid C \in \mathcal{C}\}$ of a co.co. $\mathcal{X} = (\Omega, \mathcal{C})$ is a co.co. then $\mathcal{C}$ is a scheme.

Enumerate up to an isomorphism all coherent configurations of degree $2, 3, 4$.

Triangle condition

Prove that for any triple $R, S, T \in \mathcal{C}$ it holds thatt

$$p_{RS}^{T^*}|T^*| = p_{ST}^{R^*}|R^*| = p_{TR}^{S^*}|S^*|.$$

Fiber decomposition of a co.co.

Let $\Omega_1, ..., \Omega_f$ be the complete set of fibers of a co.co. $(\Omega, \mathcal{C})$. Denote by $\mathcal{C}^{ij}, 1 \leq i, j \leq f$ the subset of $\mathcal{C}$ consisting of the relations R with $pr_1(R) = \Omega_i, pr_2(R) = \Omega_j$. Then $\mathcal{C}$ is a disjoint union of $\mathcal{C}^{ij}, 1 \leq i, j \leq f$.

Definition

Let $I \subseteq [1, f]$. Then $(\bigcup_{i \in I} \Omega_i, \bigcup_{i,j \in I} \mathcal{C}^{ij})$ is a co.co called a truncation of $(\Omega, \mathcal{C})$.

Direct sum of co.co.s

Given two co.co.s $\mathcal{X} = (\Omega, \mathcal{C}), \mathcal{X}' = (\Omega', \mathcal{C}')$ with $\Omega \cap \Omega' = \emptyset$, their direct sum $\mathcal{X} \boxplus \mathcal{X}'$ is defined as a co.co. over $\Omega \cup \Omega'$ with the set of relations $\mathcal{C} \cup \mathcal{C}' \cup \{\Omega_i \times \Omega'_j, \Omega'_j \times \Omega_i \,|\, 1 \leq i \leq f, 1 \leq j \leq f'\}$.

Isomorphisms between coherent configurations

Definition

Two coherent configuration $\mathcal{X} = (\Omega, \mathcal{C})$ and $\mathcal{X}' = (\Omega', \mathcal{C}')$ are called (combinatorially) isomorphic iff there exist bijections $f : \Omega \to \Omega', \phi : \mathcal{C} \to \mathcal{C}'$ such that

$$\forall_{\alpha,\beta\in\Omega} \;\; (\alpha, \beta) \in C \iff (\alpha^f, \beta^f) \in C^\phi.$$

The set of all isomorphisms between $\mathcal{X}$ and $\mathcal{X}'$ is denoted as $\mathrm{Iso}(\mathcal{X}, \mathcal{X}')$. The mapping ϕ is uniquely determined by f.

Abbreviation $\mathrm{Iso}(\mathcal{X}) := \mathrm{Iso}(\mathcal{X}, \mathcal{X})$,

$$\mathrm{Aut}(\mathcal{X}) = \{f \in \mathrm{Sym}(\Omega) : \;\; S^f = S \text{ for all } S \in \mathcal{C}\},$$

and $\mathrm{Aut}(\mathcal{X}) \trianglelefteq \mathrm{Iso}(\mathcal{X})$.

Examples

Trivial and discrete co.co.s.

Definition

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Proposition

A graph $\Gamma = (\Omega, E)$ is strongly regular if and only if there exist non-negative integers k, λ, μ such that

1. Γ is k-regular,
2. any pair of points connected by an edge have λ common neighbours,
3. any pair of points not connected by an edge have μ common neighbours

Exercises

Let $\Gamma = (\Omega, E)$ be a strongly regular graph with parameters v, k, λ, μ. Then

- the graph (Ω, E^c) is an s.r.g. too;
- $(k - \lambda - 1)k = (v - k - 1)\mu$;
- Γ is disconnected iff $\lambda = k - 1$ and in this case Γ is a disjoint union of K_k;
- prove that the graph $\Gamma(K)$ built from the Cayley table of a group K is strongly regular and find its parameters;
- prove that the complements to the graphs $\Gamma(\mathbb{Z}_4)$ and $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)$ are the only s.r.g. with parameters $v = 16, k = 6, \lambda = \mu = 2$ (hint: look at the neighborhood of a point.)

Examples. Permutation groups.

Let $G \leq \mathrm{Sym}(\Omega)$ be a permutation group. It acts on $\Omega \times \Omega$:

$$(\alpha, \beta)^g := (\alpha^g, \beta^g), \qquad \alpha, \beta \in \Omega,\ g \in G.$$

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Definition.

A coherent configuration $\mathcal{X}$ is called schurian if $\mathcal{X} = \mathrm{Inv}(G)$ for some group G.

Schurity problem

Given a coherent configuration $\mathcal{X}$, find whether it is schurian.

Example - Exercise

Let $G = \{\pm 1\}^3 \leq GL_3(\mathbb{R})$ and $\Omega = \{x \in \mathbb{Z}^3 \,|\, \|x\|^2 = 2\}$. Define $\mathcal{X} = (\Omega, \mathcal{C})$ where $\mathcal{C} = \Omega^2/G$.

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- what is the rank of $\mathcal{X}$?
- find the fibers and describe the basic graphs of $\mathcal{X}$;
- find homogeneous consisitiuents;
- find $\mathrm{Iso}(\mathcal{X})$ and $\mathrm{Aut}(\mathcal{X})$;

Galois correspondence.

Definition

Let $\mathcal{X} = (\Omega, \mathcal{C}), \mathcal{X}' = (\Omega, \mathcal{C}')$ be two coherent configuratios. We say that $\mathcal{X}$ is a fusion of $\mathcal{X}'$ (equivalently $\mathcal{X}'$ is a fission of $\mathcal{X}$), notation $\mathcal{X} \sqsubseteq \mathcal{X}'$ if $\mathcal{C} \sqsubseteq \mathcal{C}'$.

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- $G \leq \mathrm{Aut}(\mathrm{Inv}(G))$;

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- $H \leq G \implies \mathrm{Inv}(H) \sqsupseteq \mathrm{Inv}(G)$;
- $G \leq \mathrm{Aut}(\mathrm{Inv}(G)$;
- $\mathcal{X} \sqsubseteq \mathrm{Inv}(\mathrm{Aut}(\mathcal{X}))$.

Galois closed objects.

Definition

The group $G^{(2)} := \mathsf{Aut}(\mathsf{Inv}(G))$ is called a 2-closure of $G \leq \mathsf{Sym}(\Omega)$. A group is called 2-closed if $G = G^{(2)}$.

Definition

Given a coherent configuration $\mathcal{X} = (\Omega, \mathcal{C})$, the configuration $\mathrm{Sch}(\mathcal{X}) := \mathsf{Inv}(\mathsf{Aut}(\mathcal{X}))$ is called a Schurian closure of $\mathcal{X}$. A configuration $\mathcal{X}$ is schurian iff $\mathrm{Sch}(\mathcal{X}) = \mathcal{X}$.

Theorem

The mappings $(\mathsf{Aut}, \mathsf{Inv})$ are bijections between 2-closed subgroups of $\mathsf{Sym}(\Omega)$ and schurian coherent configurations defined on Ω.

Examples of co.co.s from block designs

Let V, B be two disjoint sets and $I \subseteq V \times B$. A triple $(V, B, I), I \subseteq V \times B$ is called a block design if

1. every block is incident k points;
2. every point is incident r blocks;
3. every pair of distinct points is incident to λ blocks.

The numbers $v := |V|, b := |B|, k, r, \lambda$ are called the parameters of the design.

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Proposition

1. $r = \lambda\frac{v-1}{k-1}$;
2. $b = \lambda\frac{v(v-1)}{k(k-1)}$.

Block Designs

Theorem (Fisher's inequality)

Let (V, B, I) is $2-(v, k, \lambda)$-design. If $\lambda < r$, then $v \leq b$

Designs with $b = v$ are called symmetric. Design is symmetric iff any two blocks are incident to the same number of points (λ). Examples of symmetric designs: projective palnes.

Exercises

1. Prove that if (V, B, I) is a $2-(v, k, \lambda)$ design, then $(V, B, V \times B \setminus I)$ is also a 2-design and find its parameters.
2. Prove that if (V, B, I) is symmetric (v, k, λ)-design, then the relations $1_V, 1_B, V^2 \setminus 1_V, B^2 \setminus 1_B, I, I^*, (V \times B) \setminus I, (B \times V) \setminus I^*$ form a c.c. on the set $V \cup B$.
3. Let $(\Omega, \mathcal{C})$ be a c.c. with two fibers, say Ω_1 and Ω_2. Prove that if $|\mathcal{C}^{11}| = |\mathcal{C}^{22}| = 2$, then $|\mathcal{C}^{12}| \leq 2$. If $\mathcal{C}^{12} = \{R_1, R_2\}$, then $(\Omega_1, \Omega_2, R_1)$ and $(\Omega_1, \Omega_2, R_2)$ are symmetric block-designs complementary to each other.

Desarguesian projective planes

Let U be a three-dimensional vector space defined over a finite field $\mathbb{F}_q$.
Point set V is the set of 1-dimensional subspaces;
Block set B is the set of 2-dimensional subspaces;
Incidence $I \subseteq V \times B$ is a usual inclusion.

Proposition

The incidence system (V, B, I) form a symmetric block design with parameters $|V| = |B| = q^2 + q + 1, k = q + 1, \lambda = 1$.

It is known as a Desarguesian projective plane of order q.