Coherent algebras and coherent configurations

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The International Workshop
G2R2
Novosibirsk, Russia, August 13 - 18, 2018

Coherent algebras.

Notation.

Let $A, B \in M_\Omega(\mathbb{F})$ be arbitrary matrices. We denote by

- AB (or $A \cdot B$) the usual matrix product;
- $A \circ B$ the Schur-Hadamard (component-wise) product, i.e. $(A \circ B)_{\alpha\beta} := A_{\alpha\beta} B_{\alpha\beta}$;
- $A^\top$ the transposed of A;
- I_Ω the identity matrix;
- J_Ω the all one matrix;
- if R is a binary relation, then $A(R)$ denotes the adjacency matrix of R.

Proposition

The algebra $(M_\Omega(\mathbb{F}), \circ)$ is a commutative associative algebra with identity J_Ω. It is isomorphic to $\mathbb{F}^n$ where $n = |\Omega|^2$.

Idempotents

Let $(\mathcal{A}, \star)$ be finite dimensional algebra over field $\mathbb{F}$.

- $e \in \mathcal{A}, e \neq 0$ is called $\star$-idempotent iff $e \star e = e$;
- idempotents e, f are orthogonal if $e \star f = f \star e = 0$;
- idempotent e is minimal if it is not a sum $e = e_1 + e_2$ of pairwise orthogonal idempotents e_1, e_2;
- a matrix $E \in M_\Omega(\mathbb{F})$ is a $\cdot$-idempotent iff it's similar to a $(0,1)$-diagonal matrix;
- a matrix $E \in M_\Omega(\mathbb{F})$ is a $\circ$-idempotent iff it's $(0,1)$-matrix, that is $E = A(S)$ is the adjacency matrix of some $S \subseteq \Omega^2$, where

$$A(S)_{\alpha\beta} := \begin{cases} 1 & (\alpha,\beta) \in S; \\ 0 & (\alpha,\beta) \notin S \end{cases}$$

Exercises.

1. Prove that $A(S) \circ A(T) = A(S \cap T)$ for any pair of $S, T \subseteq \Omega^2$.
2. Prove that if $S_1, \dots, S_k \subseteq \Omega^2$ are pairwise disjoint, then $A(S_1) + \dots + A(S_k) = A(S_1 \cup \dots \cup S_k)$.

Coherent (cellular) algebras.

Definition.

A subspace $\mathcal{A} \leq M_\Omega(\mathbb{F})$ is called a coherent (or cellular) algebra if it contains I_Ω, J_Ω and is closed with respect to $\cdot, \circ, ^\top$.

Examples

- $\langle I_\Omega, J_\Omega \rangle$;
- $M_\Omega(\mathbb{F})$.

Theorem

Let $\mathcal{X} = (\Omega, \mathcal{C})$ be a coherent configuration. Then the linear span $\mathbb{F}[\mathcal{C}] := \langle A(C) \rangle_{C \in \mathcal{C}}$ is a coherent algebra of dimension $|\mathcal{C}|$. It is called the adjacency (or Bose-Mesner) algebra of $\mathcal{X}$.

Proof

Recall that a pair $\mathcal{X} = (\Omega, \mathcal{C}), \mathcal{C} \vdash \Omega^2$ is a co.co. if it satisfies the following conditions:

1. $1_\Omega \in \mathcal{C}^\cup$;
2. $\mathcal{C}^* = \mathcal{C}$;
3. for all $R, S, T \in \mathcal{C}$ the number $p_{RS}^T = |R(\alpha) \cap S^*(\beta)|$ does not depend on the choice of $(\alpha, \beta) \in T$.

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$\mathcal{C} \vdash \Omega^2 \implies \mathbb{F}[\mathcal{C}] \circ \mathbb{F}[\mathcal{C}] \subseteq \mathbb{F}[\mathcal{C}]$ and $\sum_{S \in \mathcal{C}} A(S) = J_\Omega \in \mathbb{F}[\mathcal{C}]$.

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$\mathcal{C}^* = \mathcal{C} \implies \mathbb{F}[\mathcal{C}]^\top = \mathbb{F}[\mathcal{C}]$.

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$1_\Omega \in \mathcal{C}^\cup \implies I_\Omega \in \mathbb{F}[\mathcal{C}]$.
$\mathcal{C}^* = \mathcal{C} \implies \mathbb{F}[\mathcal{C}]^\top = \mathbb{F}[\mathcal{C}]$.
The third condition implies that

$$A(S)A(T) = \sum_{R \in \mathcal{C}} p_{ST}^R A(R)$$

Therefore, $\mathbb{F}[\mathcal{C}] \cdot \mathbb{F}[\mathcal{C}] \subseteq \mathbb{F}[\mathcal{C}]$. $\square$

Adjacency algebra of a coherent configuration.

1. The algebra $\mathbb{F}[\mathcal{C}]$ is called the adjacency algebra of the co.co. $(\Omega, \mathcal{C})$.
2. $\mathbb{F}[\mathcal{C}]$ is a coherent subalgebra of $M_\Omega[\mathbb{F}]$.
3. The basis $A(C), C \in \mathcal{C}$ is called the (first) standard basis of $\mathbb{F}[\mathcal{C}]$. It consists of minimal $\circ$-idempotents.
4. The intersection numbers $p_{R,S}^{T}$ of the co.co. become the structure constants of the adjacency algebra $\mathbb{F}[\mathcal{C}]$ in its standard basis.

Intersection algebra

A regular representation of the adjacency algebra is called the intersection algebra of a co.co. The intersection matrix $(M_R)_{T,S} = (p_{RS}^T), R \in \mathcal{C}$ presents the matrix of $A(R)$ in a regular representation in the standard basis $A(R), R \in \mathcal{C}$.

Example

Let (Ω, R) be a strongly regular graph with parameters (v, k, λ, μ). The partition $1_\Omega, R, R^c = \Omega^2 \setminus R \setminus 1_\Omega$ of Ω^2 form a homogeneous co.co. Its intersection matrices have the following form: $M_{1_\Omega} = I_3$,

$$M_R = \begin{bmatrix} 0 & k & 0 \\ 1 & \lambda & k-\lambda-1 \\ 0 & \mu & k-\mu \end{bmatrix}, M_{R^c} = \begin{bmatrix} 0 & 0 & v-k-1 \\ 0 & k-\lambda-1 & v-2k+\lambda \\ 1 & k-\mu & v-2k+\mu-2 \end{bmatrix}$$

Properties of intersection numbers

Let $\mathcal{X} = (\Omega, \mathcal{C})$ be a co.co. with fibers $\Omega_1, ..., \Omega_f$. Then

1. $\forall_{S \in \mathcal{C}}\ |S| = k_S |pr_1(S)| = k_{S^*} |pr_2(S)| = |S^*|$;
2. $\forall_{S \in \mathcal{C}^{ij}} |S| \equiv 0 \bmod(\mathrm{lcm}(|\Omega_i|, |\Omega_j|)$, and $|\mathcal{C}^{ij}| \leq \gcd(|\Omega_i|, |\Omega_j|)$;
3. $\sum_{S \in \mathcal{C}^{ij}} k_S = |\Omega_j|$;
4. $p_{R,S}^T = p_{S^*,R^*}^{T^*}$;
5. $p_{R,S}^T |T| = p_{R^*T}^S |S| = p_{T,S^*}^R |R|$ (the triangle condition);
6. $\sum_S p_{RS}^T = k_R$ if $pr_1(R) = pr_1(T)$ and zero otherwise;
7. $\sum_T p_{RS}^T k_T = k_R k_S$.

Properties of intersection numbers of homogeneous co.co.s

1. $\forall_{S\in\mathcal{C}}\ k_S = k_{S^*}$;
2. $\sum_{S\in\mathcal{C}} k_S = |\Omega|$;
3. $p_{R,S}^{T} k_T = p_{T^*R}^{S^*} k_S = p_{T,S^*}^{R} k_R$ (the triangle condition);
4. $\forall_{R,S,T\in\mathcal{C}}\ \ p_{R,S}^{T} k_T \equiv 0 \bmod(\mathrm{lcm}(k_R, k_S))$, and $|\mathrm{Supp}(RS)| \leq \gcd(k_R, k_S)$;

From coherent algebras to coherent configurations

Theorem

Every coherent algebra $\mathcal{A} \leq M_\Omega(\mathbb{F})$ has a unique basis consisting of minimal $\circ$-idempotents which are pairwise orthogonal. If $\mathrm{char}(\mathbb{F}) = 0$ then $\mathcal{A}$ is the adjacency algebra of a uniquely determined coherent configuration.

Since $\mathcal{A}$ is a $\circ$-subalgebra of $(M_\Omega(\mathbb{F}), \circ) \cong (\mathbb{F}^n, \circ)$, $n = |\Omega|^2$, we start with the following

Lemma

Let $\mathcal{A}$ be a k-dimensional subalgebra of $(\mathbb{F}^n, \circ)$. Then there exists a unique basis $A_1, \ldots, A_k$ of $\mathcal{A}$ consisting of minimal, pairwise orthogonal, $\circ$-idempotents. Each $\circ$-idemoptent of $\mathcal{A}$ is a $(0, 1)$-linear combination of $A_1, \ldots, A_k$ and $A_1 + \ldots + A_k$ is the unit of $\mathcal{A}$.

Proof of the Theorem

Let $A_1, \dots, A_r$ be a basis of $M_\Omega(\mathbb{F})$ consisting of minimal $\circ$-idempotents. Then

- $A_i = A(R_i), R_i \subseteq \Omega^2$;
- $i \neq j \implies A_i \circ A_j = 0 \implies R_i \cap R_j = \emptyset$;
- $\sum_{i=1}^{k} A_i = J_\Omega \implies \bigcup_i R_i = \Omega^2$;
- $I_\Omega = \sum_i A_i \implies 1_\Omega = \bigcup_i R_i$;
- $A_i^\top \in \{A_1, \dots, A_r\} \implies R_i^* \in \{R_1, \dots, R_r\}$;
- $A_i A_j = \sum_k p_{ij}^k A_k$ for some $p_{ij}^k \in \mathbb{F}$
- $(A_i A_j)_{\alpha\beta} = p_{ij}^k$ where k is defined by $(\alpha, \beta) \in R_k$;
- if $\mathrm{char}(\mathbb{F}) = 0$, then
  $(A_i A_j)_{\alpha\beta} = |R_i(\alpha) \cap R_j^*(\beta)| \implies |R_i(\alpha) \cap R_j^*(\beta)| = p_{ij}^k$.

Isomorphisms between coherent algebras

Definition

Given two coherent algebras $\mathcal{A} \leq M_\Omega(\mathbb{F}), \mathcal{A}' \leq M_{\Omega'}(\mathbb{F})$, a linear bijection $L : \mathcal{A} \to \mathcal{A}'$ is called an (algebraic) isomorphism iff

- $L(XY) = L(X)L(Y)$;
- $L(X \circ Y) = L(X) \circ L(Y)$;
- $L(X^\top) = L(X)^\top$.

Exercise: Prove that $L(I_\Omega) = I_{\Omega'}, L(J_\Omega) = J_{\Omega'}$.

Isomorphisms between coherent algebras

Proposition

Let $L : \mathbb{F}[\mathcal{C}] \to \mathbb{F}[\widetilde{\mathcal{C}}]$ be an algebraic isomorphism between the adjacency algebras of co.co.s $\mathcal{C}$ and $\widetilde{\mathcal{C}}$. Then there exists a bijection $\varphi : \mathcal{C} \to \widetilde{\mathcal{C}}$ such that $L(A(C)) = A(C^\varphi)$ and $c_{RS}^{T} = \widetilde{c}_{R^\varphi S^\varphi}^{T^\varphi}$. Vice versa, any bijection $\varphi : \mathcal{C} \to \widetilde{\mathcal{C}}$ satisfying the above equations extends uniquely up to an algebraic isomorphism between $\mathbb{F}[\mathcal{C}]$ and $\mathbb{F}[\widetilde{\mathcal{C}}]$. We'll call it an algebraic isomorphism between the co.co.s.

Properties of algebraic isomorphisms

Proposition

Let $\varphi : \mathcal{C} \to \widetilde{\mathcal{C}}$ be an algebraic isomorphism between the co.co.s $\mathcal{X} = (\Omega, \mathcal{C})$ and $\widetilde{\mathcal{X}} = (\widetilde{\Omega}, \widetilde{\mathcal{C}})$. Then

- $(\mathrm{Supp}_{\mathcal{C}}(RS))^{\varphi} = \mathrm{Supp}_{\mathcal{C}'}(R^{\varphi} S^{\varphi})$ for any $R, S \in \mathcal{C}^{\cup}$
- for each fiber Δ of $\mathcal{X}$ there exists a unique fiber Δ' of $\mathcal{X}'$ such that $(1_{\Delta})^{\varphi} = 1_{\Delta'}$, that is φ induces a bijection between fibres;
- $S \in \mathcal{C} \implies pr_1(S^{\varphi}) = pr_1(S)^{\varphi}, pr_2(S^{\varphi}) = pr_2(S)^{\varphi}$;
- $|\Delta^{\varphi}| = |\Delta|$ for any $\Delta \in \Phi(\mathcal{X})$;
- $|C^{\varphi}| = |C|$ and $k_{C^{\varphi}} = k_C$ hold for each $C \in \mathcal{C}$;

Isomorphisms between coherent algebras

Proposition

For each $f \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}')$ the mapping f^* is an algebraic isomorphism between the configurations. In this case we say that f^* is an algebraic isomorphism induced by a combinatorial one.

All algebraic automorphisms of a co.co. $\mathcal{X} = (\Omega, \mathcal{C})$ form a group (a subgroup of $\mathrm{Sym}(\mathcal{C})$) denoted as $\mathrm{Alg}(\mathcal{X})$ or $\mathrm{Alg}(\mathcal{C})$. Notice that $\mathrm{Iso}(\mathcal{X})/\mathrm{Aut}(\mathcal{X}) \hookrightarrow \mathrm{Alg}(\mathcal{X})$.

Proposition

Let $A \leq \mathrm{Alg}(\mathcal{X})$. Then the subspace

$$\mathbb{Q}[\mathcal{C}]^A := \{x \in \mathbb{Q}[\mathcal{C}] \mid \forall_{a \in A}\ x^a = x\}$$

is a coherent algebra. The corresponding coherent conguration is denoted as $\mathcal{C}^A$. It is called an algebraic fusion of $\mathcal{C}$.

Coherent closure.

Proposition. Let $\mathcal{X} = (\Omega, \mathcal{C})$ and $\mathcal{X}' = (\Omega, \mathcal{C}')$ be coherent configurations. Then.

- $\mathbb{F}[\mathcal{C}] \subseteq \mathbb{F}[\mathcal{C}'] \iff \mathcal{C} \sqsubseteq \mathcal{C}'$;
- $\mathbb{F}[\mathcal{C}] \cap \mathbb{F}[\mathcal{C}'] = \mathbb{F}[\mathcal{C} \wedge \mathcal{C}'] \implies \mathcal{C} \wedge \mathcal{C}'$ is a coherent configuration.

Notice that the sum of coherent algebras is not necessarily coherent algebra.

Proposition

Let $A_1, \ldots, A_m \in M_\Omega(\mathbb{F})$ be an arbitrary sequence of matrices. The intersection of all coherent algebras containing $A_1, \ldots, A_m$ is a coherent algebra too, called the coherent closure of $A_1, \ldots, A_m$ and denoted as $\langle\langle A_1, \ldots, A_m \rangle\rangle$.

Computing coherent closure by Weisfeiler-Leman algortihm.

Definition

Given a sequence $M_1, ..., M_k \in M_\Omega(\mathbb{F})$ of matrices. We define a partititon $\mathcal{P}(M_1, ..., M_k)$ of Ω^2 via the following equivalence relation

$$(\alpha, \beta) \sim (\gamma, \delta) \iff \forall_{1 \leq i \leq k} \ (M_i)_{\alpha\beta} = (M_i)_{\gamma\delta}$$

Proposition

If $\mathcal{X} = (\Omega, \mathcal{C})$ is a coherent configuration, then $\mathcal{P}(M_1, ..., M_k) \sqsubseteq \mathcal{C}$ for any tuple $M_1, ..., M_k \in \mathbb{F}[\mathcal{C}]$.

Operation Ы

Let $\mathcal{C}$ be a partition of Ω^2.

- For all $(\alpha, \beta) \in \Omega \times \Omega$ and $R, S \in \mathcal{C}$ find the number

$$c(\alpha, \beta; R, S) = |R(\alpha) \cap S^*(\beta)|.$$

- Build a new partition Ы$(\mathcal{C})$ by putting (α, β) and (α', β') to the same class of Ы(S) if $|R(\alpha) \cap S^*(\beta)| = |R(\alpha') \cap S^*(\beta')|$ for all $R, S \in \mathcal{C}$.

Computing coherent closure by WL algortihm.

- **Input:** $A_1, \ldots, A_k \in M_\Omega(\mathbb{F})$,
- **Output:** the coherent closure $\langle\langle A_1, \ldots, A_k \rangle\rangle$;
- Compute $\mathcal{S}_0 := \mathcal{P}(A_1, \ldots, A_k, A_1^\top, \ldots, A_k^\top, I_\Omega)$;
- Starting from $\mathcal{S} := \mathcal{S}_0$ apply WL-stabilization procedure $\mathcal{S} \to \text{ЫI}(\mathcal{S})$ until $\mathcal{S} = \text{ЫI}(\mathcal{S})$.

Theorem

The WL-algorithm produces the coherent closure of the matrices $A_1, \ldots, A_k$.

Application to the GIP

Proposition

Let $\vec{\mathcal{S}} = (S_1, ..., S_m)$ and $\vec{\mathcal{T}} = (T_1, ..., T_m)$ be ordered partitions of Ω^2 and Δ^2 resp. WL-algorithm constructs ordered c.c.s $\langle\langle\vec{\mathcal{S}}\rangle\rangle = (P_1, ..., P_k)$ and $\langle\langle\vec{\mathcal{T}}\rangle\rangle = (Q_1, ..., Q_\ell)$ s.t. if there exists an isomorphism $f : \Omega \to \Delta$ such that $S_i^f = T_i$, then $k = \ell$ and $P_i^f = Q_i, i = 1, ..., k$. In particular, the mapping $P_i \mapsto Q_i$ is an algebraic isomorphism between the co.co.s $\langle\langle\mathcal{S}\rangle\rangle$ and $\langle\langle\mathcal{T}\rangle\rangle$.

Reformulation of the GIP

Given an algebraic isomorphism between two coherent configurations $\mathcal{S}$ and $\mathcal{T}$. Find whether it is induced by a combinatorial one.

Association schemes.

Recall that a pair $\mathcal{X} = (\Omega, S)$ where $\mathcal{S} \vdash \Omega^2$ is an association scheme (=homogeneous coherent configuration) iff $\mathcal{S}$ is a c.c. with $1_\Omega \in \mathcal{S}$.

The intersection numbers of $\mathcal{X}$ are

$$\forall_{S,R,T \in \mathcal{S}} \ \forall_{(\alpha,\beta) \in T} \ |S(\alpha) \cap R^*(\beta)| = p_{RS}^T.$$

The number $k_S := p_{SS^*}^{1_\Omega} = |S(\omega)|$ is called the valency of S.

Elementary properties of schemes.

Proposition

- each relation $R \in \mathcal{S}^{\cup}$ is regular;
- the set $\mathcal{S}^{\cup}$ is closed w.r.t. boolean operations;
- $1_\Omega, \Omega^2 \in \mathcal{S}^{\cup}$;
- $(\mathcal{S}^{\cup})^* = \mathcal{S}^{\cup}$;
- $\mathcal{S}$ is closed w.r.t. relational product;
- for any $S \in \mathcal{S}$ there exists m s.t. $1_\Omega \in S^m$;
- $S \in \mathcal{S}^{\cup} \implies S^+ = \cup_{i=1}^{\infty} S^i \in \mathcal{S}^{\cup}$;
- $S \in \mathcal{S}^{\cup} \implies S^+$ is an equivalence relation on Ω

Elementary properties of schemes.

Definition

A scheme is called symmetric (antisymmetric) if every $S \neq 1_\Omega$ is symmetric (anti-symmetric, resp.). A scheme is called commutative if its adjacency algebra is commutative.

Proposition

A symmetric scheme is always commutative.

Proposition

Let $(\Omega, \mathcal{S})$ be a scheme. Then

- $\sum_{S \in \mathcal{S}} k_S = |\Omega|$;
- $2 \leq |\mathcal{S}| \leq |\Omega|$;
- $|\mathcal{S}| = 2 \iff \mathcal{S} = \{1_\Omega, \Omega^2 \setminus 1_\Omega\}$;

Primitive and Imprimitive schemes

Definition

A scheme $(\Omega, \mathcal{S})$ is called imprimitive if $\mathcal{S}^{\cup}$ contains a non-trivial and non-discrete equivalence relation E. The equivalence classes $E(\omega), \omega \in \Omega$ form a partition of Ω called imprimitivity system of $\mathcal{S}$.

Proposition

If $E \in \mathcal{S}^{\cup}$ is an equivalence. Then $|\Omega/E| \cdot k_E = |\Omega|$.

Proposition

The following are equivalent

- $\mathcal{S}$ is imprimitive;
- $\exists\ S \in \mathcal{S}, S \neq 1_\Omega$ s.t $S^+ \neq \Omega^2$;
- $\exists\ \mathcal{T} \subset \mathcal{S}, 1 < |\mathcal{T}| < |\mathcal{S}|$ s. t. $\langle A(T) \rangle_{T \in \mathcal{T}}$ is a subalgebra of $\mathbb{Q}[\mathcal{S}]$

Schurian association schemes

Recall that a co.co. $(\Omega, \mathcal{S})$ is schurian iff $\mathcal{S} = \Omega^2/G$ for some $G \leq \mathrm{Sym}(\Omega)$.

- $(\Omega, \mathcal{S})$ is an association scheme iff G is transitive on Ω;
- $S(\omega)$ is an orbit of G_ω for each $\omega \in \Omega$ and $S \in \mathcal{S}$;
- the mapping $S \mapsto \{g \in G \,|\, \omega^g \in S(\omega)\}$ is a bijection between the basic relations of $\mathcal{S}$ and double cosets of G_ω.
- the above mapping induces an isomorphism between the adjacency algebra $\mathbb{F}[\mathcal{S}]$ and the Hecke algebra $\mathbb{F}(G_\omega \backslash G / G_\omega)$;
- G is primitive iff $(\Omega, \Omega^2/G)$ is primitive;
- G is primitive iff G_ω is maximal.

Thin/regular association schemes

An association scheme $\mathcal{X} = (\Omega, \mathcal{C})$ is called thin/regular if $k_S = 1$ for all $S \in \mathcal{C}$.
Given a group G, define two partitions G_L and G_R of $G \times G$ as follows:
$G_L = \{g_L \mid g \in G\}$ where $g_L = \{(x, g^{-1}x) \mid x \in G\}$;
$G_R = \{g_R \mid g \in G\}$ where $g_R = \{(x, xg) \mid x \in G\}$.

Proposition

- both (G, G_L) and (G, G_R) are association schemes;
- (G, G_L) and (G, G_R) are isomorphic;
- $G_R = \mathrm{Aut}((G, G_L))$, $G_L = \mathrm{Aut}((G, G_R))$;

Association schemes

Exercise

Prove that every regular scheme is isomorphic to (G, G_R) for some group G.

Definition

A fusion of a regular scheme (G, G_L) is called a Cayley scheme over G. If G is abelian, then it is often called a translation scheme.

Theorem

An association scheme $\mathcal{X} = (\Omega, \mathcal{S})$ is a Cayley scheme over G iff $\mathrm{Aut}(\mathcal{X})$ contains a regular subgroup isomorphic to G.

Schur rings

Fusions of a regular scheme (G, G_L) are in one-to-one correspondence with partitions $\mathcal{S} \vdash G$ which satisfy the following conditions:

Schur rings

Fusions of a regular scheme (G, G_L) are in one-to-one correspondence with partitions $\mathcal{S} \vdash G$ which satisfy the following conditions:

1. $\{e\} \in \mathcal{S}$;
2. $S \in \mathcal{S} \implies S^{(-1)} := \{s^{-1} \mid s \in S\} \in \mathcal{S}$;
3. $\forall_{R,S,T \in \mathcal{S}}$ the number $|\{(x, y) \in R \times S \mid xy = t\}|$ does not depend on a choice of $t \in T$

A partition satisfying the above conditions is called a Schur partition of G.

Proposition

The partition $\mathcal{S} \vdash G$ which satisfies (1)-(2) is a Schur partition iff $\langle \underline{S} \rangle_{S \in \mathcal{S}}$ is a subalgebra of a group algebra $\mathbb{Z}[G]$.

Subalgebras of this type are called Schur rings/algebras over G.

Schemes of rank three

Proposition

If $(\Omega, \mathcal{S} = \{S_0, S_1, S_2\})$ is a scheme of rank three, then it is either symmetric (i.e. $S_1^* = S_1, S_2^* = S_2$) or anti-symmetric (i.e. $S_1^* = S_2, S_2^* = S_1$).

If the scheme is antisymmetric, then its non-identical basic relations are known as doubly regular tournaments.

Theorem (Reid & Brown)

An antisymmetric scheme of order v exists iff there exists a skew-symmetric Hadamard matrix of order $v+1$.

Conjecture

Is it true that there exists a skew-symmetric Hadamard matrix of order N for every N divisible by 4?

C. Koukouvinos, S. Stylianou, On skew-Hadamard matrices, DM (308), 2008.

Symmetric schemes of rank three

Let $(\Omega, \mathcal{S} = \{S_0, S_1, S_2\})$ be a symmetric association scheme. Then both (Ω, S_1) and (Ω, S_2) are strongly regular graphs.

Symmetric schemes of rank three

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Proposition

Let $\Gamma = (\Omega, E)$ be a simple graph and A its adjacency matrix. TFAE

1. Γ is a s.r.g.;
2. there exists integers $k > 0, \lambda, \mu \geq 0$ s.t.

$$|E(\alpha) \cap E(\beta)| = \begin{cases} k & \alpha = \beta; \\ \lambda & (\alpha, \beta) \in E; \\ \mu & (\alpha, \beta) \notin E \end{cases}$$

3. there exists integers $k > 0, \lambda, \mu \geq 0$ s.t. $A^2 = kI_\Omega + \lambda A + \mu(J - A - I)$.

Concrete examples

Question

Does there exist a s.r.g. with given parameters (v, k, λ)?

Example 1

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Example 2

Does there exist an s.r.g. with parameters $(17, 4, 0)$?

Concrete examples

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Does there exist a s.r.g. with given parameters (v, k, λ)?

Example 1

Does there exist an s.r.g. with parameters $(17, 4, 1)$?
No, it doesn't because $\mu = \frac{k(k-1-\lambda)}{v-1-k} = 8/12 \notin \mathbb{Z}$.

Example 2

Does there exist an s.r.g. with parameters $(17, 4, 0)$?
In this case $\mu = \frac{k(k-1-\lambda)}{v-1-k} = 1$ Computation of intersection numbers of association scheme yields us

$$M_1 = \begin{bmatrix} 0 & 4 & 0 \\ 1 & 0 & 3 \\ 0 & 1 & 3 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 0 & 12 \\ 0 & 3 & 9 \\ 1 & 3 & 8 \end{bmatrix}$$

Eigenvalues of a strongly regular graph

Theorem

Let Γ be a connected srg with parameters v, k, λ, μ.. Then its adjacency matrix A has three eigenvalues $s < r < k$, where

$$\begin{aligned} r &= \frac{(\lambda-\mu)+\sqrt{(\lambda-\mu)^2+4(k-\mu)}}{2} \\ s &= \frac{(\lambda-\mu)-\sqrt{(\lambda-\mu)^2+4(k-\mu)}}{2}. \end{aligned}$$

An eigenvalue k has multiplicity 1. The multiplicities of two others, r and s, are given by the following formulae

$$\begin{aligned} f &= -\frac{(v-1)s+k}{r-s}, \\ g &= \frac{k+(v-1)r}{r-s} \end{aligned}$$

where f is the multiplicity of r while g is a multiplicity of s.

Moore graphs

Definition

A connected strongly regular graph of a maximal girth is called a Moore graph ($g = 5$).

Proposition

A Moore graphs has parameters $(k^2 + 1, k, 0, 1)$, $k \in \mathbb{N}$.

Theorem

If Moore graph Γ of degree k exists, then $k \in \{2, 3, 7, 47\}$.

1. $k = 2 \implies \Gamma \cong C_5$;
2. $k = 3 \implies \Gamma \cong$ the Petersen graph;
3. $k = 7 \implies \Gamma \cong$ the Hoffman-Singleton graph;
4. $k = 57$ is open.