Representation theory of coherent configurations/algebras

M. Muzychuk

Ben-Gurion University of the Negev,
Israel

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Linear representations of algebras

Let $\mathcal{A}$ be a (finite dimensional) unital associative algebra over the field $\mathbb{F}$; V a f.d. vector space over $\mathbb{F}$.

A homomorphism $\Delta : \mathcal{A} \to \mathrm{End}(V)/M_n(\mathbb{F})$ is called a (linear) representation of $\mathcal{A}$ over $\mathbb{F}$. The vector space V becomes a left $\mathcal{A}$-module via $A \cdot v = \Delta(A) \cdot v$; $\dim(V)$ is called the dimension of V.

A linear function $\chi : \mathcal{A} \to \mathbb{F}$ defined via $\chi(A) = tr(\Delta(A))$ is called the character of Δ afforded by V.

Two representations $\Delta : \mathcal{A} \to \mathrm{End}(V), \Delta' : \mathcal{A} \to \mathrm{End}(V')$ are equivalent/isomorphic iff there exists a linear bijection $L : V \to V'$ s.t.

$$\forall_{A\in\mathcal{A}} \, \forall_{v\in V} \; L(\Delta(A)(v)) = \Delta'(A)(L(v)).$$

Decomposition of representations

1. Given an $\mathcal{A}$-module V, a subspace $W \subseteq V$ is an $\mathcal{A}$-submodule/subrepresentation iff $\mathcal{A}W \subseteq W$.
2. Submodule is proper iff $W \neq V$ and non-trivial if $W \neq \{0\}$.
3. The module V is a direct sum of its submodules $V_1, \ldots, V_k \subseteq V$ if V is a direct sum of $V_1, \ldots, V_k$ as a vector space.
4. An $\mathcal{A}$-representation/module is irreducible/simple if it does not contain non-trivial proper subrepresentations/submodules.

Example

$\mathcal{A} = M_n(\mathbb{F}), V = \mathbb{F}^n$ (column space)

Semisimple algebras

A module M over algebra $\mathcal{A}$ is semisimple if it is a direct sum of simple modules.
An algebra $\mathcal{A}$ over $\mathbb{F}$ is called semisimple if its left regular module is semisimple $\mathcal{A}$-module.

Definition

A subspace $\ker(\Delta) := \{a \in A \,|\, \Delta(a) = O_V\}$ is called the kernel of Δ. A representation/module is faithful if $\ker(\Delta)$ is trivial.

Theorem

An algebra is semisimple iff it has a faithful semisimple module.

In what follows $\mathcal{A} \subseteq M_\Omega(\mathbb{C})$ is an arbitrary subalgebra with $I_\Omega \in \mathcal{A}$. The vector space $\mathbb{C}^\Omega$ is a faithful $\mathcal{A}$-module, called the standard module. The character of the standard module is denoted by τ.

Standard module and Hermitian form

We define a Hermitian form on the standard module $V = \mathbb{C}^\Omega$ via $(f, g) := \sum_{\omega\in\Omega} f(\omega)\overline{g(\omega)}$. The "induced" form on $M_\Omega(\mathbb{C})$ is defined via

$$(A, B) := \sum_{\beta\in\Omega}(A^\beta, B^\beta) = \sum_{\alpha,\beta\in\Omega} A_{\alpha\beta}\overline{B_{\alpha\beta}} = \mathrm{tr}(A\overline{B}^\top).$$

In what follows we abbreviate $\overline{A}^\top$ as A^*.

Proposition

- $(A, B) = \tau(AB^*)$;
- $(Au, v) = (u, A^*v)$;
- $(AB, C) = (B, A^*C) = (A, CB^*)$

Semisimplicity Theorem

Theorem

Let $\mathcal{A} \subseteq M_\Omega(\mathbb{C})$ be a matrix subalgebra closed w.r.t. Hermitian transposition $A \mapsto A^*$. Then $\mathcal{A}$ is semisimple.

Proof.

Main idea: if $U \leq V$ is $\mathcal{A}$-invariant, then $U^\perp$ is $\mathcal{A}$-invariant.
V is an orthogonal sum of minimal $\mathcal{A}$-invariant subspaces. $\square$

Artin - Wedderburn Theorem

Theorem (Artin - Wedderburn)

A finite dimensional semisimple algebra $\mathcal{A}$ over the field $\mathbb{F}$ is isomorphic to a direct sum of matrix algebras $M_{n_i}(D_i)$ where D_i is a division algebra over $\mathbb{F}$.

If $\mathbb{F}$ is algebraically closed, then $D_i = \mathbb{F}$.

Corollary

Let $\mathcal{A} \subseteq M_\Omega(\mathbb{C})$ be a subalgebra closed w.r.t. Hermitian transposition. Then $\mathcal{A} \cong \oplus_i M_{n_i}(\mathbb{C})$. In particular, $\dim(\mathcal{A}) = \sum_i n_i^2$.

Corollary

If $\dim(\mathcal{A}) \leq 3$, then $\mathcal{A}$ is commutative.

Theorem (Higman)

Let $\mathcal{A}$ be a coherent algebra of dimension at most 5. Then $\mathcal{A}$ is commutative.

An Example: a strongly regular decomposition of a complete graph

Theorem

Let $K_n = S_1 \cup S_2 \cup S_3$ be a decomposition of a complete graph into a disjoint union of strongly regular graphs. Then $\mathcal{C} = \{1_\Omega, S_1, S_2, S_3\}$ is an association scheme.

Proof.

Let A_i denote an adjacency matrix of (Ω, S_i). Define $\mathcal{A}_0 = \langle I, A_1, A_2, A_3 \rangle$. We have to show that $\mathcal{A}_0$ is subalgebra of $M_\Omega(\mathbb{C})$.

- $\mathcal{A}_0^\top = \mathcal{A}_0$.
- $J = I + A_1 + A_2 + A_3 \implies \mathcal{A}_0 = \langle I, A_1, A_2, J \rangle$.
- $A_i^2 = k_i I + \lambda A_i + \mu_i (J - A_i - I) \in \mathcal{A}_0$.
- $A_i A_j + A_j A_i = (A_i + A_j)^2 - A_i^2 - A_j^2 \equiv (A_i + A_j)^2 (\mathrm{mod}\ \mathcal{A}_0) \equiv (J - I - A_k)^2 (\mathrm{mod}\ \mathcal{A}_0) \equiv 0 (\mathrm{mod}\ \mathcal{A}_0)$.

End of the proof

- $\mathcal{A} := \mathcal{A}_0 + \langle A_1 A_2 \rangle = \mathcal{A}_0 + \langle A_2 A_1 \rangle$.
- $\mathcal{A} = \langle I, A_1, A_2, A_1 A_2, J \rangle$ is closed w.r.t. matrix multiplication.
- If $\mathcal{A}_0 = \mathcal{A}$, i.e. $A_1 A_2 \in \mathcal{A}_0$, then we are done.
- Otherwise, $\mathcal{A}$ is a 5-dimensional non-commutative hermitian closed matrix algebra.
- $\mathcal{A}$ has 2 irreps Δ_0, Δ_1 with $\Delta_0(I) = 1, \Delta_1(I) = 2$.
- $\Delta_0(A_i) = k_i, m_0 = 1$.
- Let $\tau(A), A \in M_\Omega(\mathbb{C})$ denote the trace of A. Then $\tau = \chi_0 + m_1\chi_1 \implies \tau(A) = \chi_0(A) + m_1\chi_1(A)$ for all $A \in \mathcal{A}$.
- $A = I \implies |\Omega| = 1 + 2m_1$,
  $A = A_i \implies 0 = k_i + m_1\chi_1(A_i) \implies k_i = m_1 u_i, u_i \in \mathbb{Z}$.
- $1 + 2m_1 = |\Omega| = 1 + k_1 + k_2 + k_3 \geq 1 + 3m_1$. A contradiction.

An alternative proof based on graph eigenvalues was proposed by E. van Dam.

Semisimplicity

- the standard module $V := \mathbb{C}^{\Omega}$ has a decomposition into a direct sum of irreducible submodules: $V = m_1 V_1 \oplus ... m_\ell V_\ell$, where V_i's are pairwise non-isomorphic. This provides a decomposition of the standard character into the sum of irreducibles: $\tau = \sum_{i=1}^{\ell} m_i \chi_i$.
- $\mathcal{A} \cong M_{n_1}(\mathbb{C}) \oplus ... \oplus M_{n_\ell}(\mathbb{C})$.
- $\Delta_i : \mathcal{A} \mapsto M_{n_i}(\mathbb{C})$ and $\Delta_i(\mathcal{A}) = M_{n_i}(\mathbb{C})$.
- $I_\Omega = E_1 + ... + E_\ell$ where E_i is a central primitive idempotent of $\mathcal{A}$ corresponding to Δ_i, i.e. $\Delta_i(E_j) = \delta_{i,j} I_{V_i}$ and $\chi_j(E_i A) = \delta_{i,j}\chi_j(A)$.
- $E_i\mathcal{A}$ is a two-sided ideal of $\mathcal{A}$ isomorphic to $M_{n_i}(\mathbb{C})$.

Central primitive idempotents

Given a function $\theta : \mathcal{A} \to \mathbb{C}$, we abbreviate $\theta(A(R))$ as $\theta(R)$.

Proposition

Let E_i be a primitive c. idempotent corresponding to Δ_i. Then

1. $E_i = m_i \sum_R |R|^{-1} \chi(R^*) A(R)$;
2. If $pr_1(R) \neq pr_2(R)$, the $\chi_i(R) = 0$.

Proof.

Write $E_i = \sum_{R \in \mathcal{C}} x_R A(R)$.
$\tau(E_i A(R^*)) = \sum_{i=1}^{\ell} m_i \chi_i(E_i A(R^*)) = m_i \chi_i(R^*)$.
On the other hand, $\tau(E_i A(R^*)) = \tau(E_i A(R)^*) = (E_i, A(R)) = x_R |R| \Rightarrow x_R = m_i \frac{\chi_i(R^*)}{|R|}$. □

Principal representation

Proposition

Let $(\Omega, \mathcal{C})$ be a co.co. with f fibers: $\Omega_1, \ldots, \Omega_f$. Then

1. If $R \in \mathcal{C}$ then $A(R)\underline{\Omega_j} = k_R \underline{\Omega_i}$ where $\Omega_i = pr_1(R), \Omega_j = pr_2(R)$;
2. The linear span of $\underline{\Omega_i}, i = 1, \ldots, f$ is an irreducible $\mathcal{A}$-submodule. The corresponding representation is called principal (notation Δ_0, $\chi_0 := \mathrm{tr}(\Delta_0)$).
3. $pr_1(R) = \Omega_i, pr_2(R) = \Omega_j \implies \Delta_0(R)_{i'j'} = \delta_{i'i}\delta_{j'j}k_R$ and zero otherwise;
4. $\dim(\Delta_0) = f$, $m_0 = 1$;
5. $E_0 = \sum_{j=1}^{d} \frac{1}{|\Omega_j|} J_{\Omega_j}$.

If $(\Omega, \mathcal{C})$ is homogeneous, then Δ_0 is one-dimensional representation of multiplicity 1.

Higman's Theorem

The principal irrep is a unique irrep of $\mathcal{A}$ iff $(\Omega, \mathcal{C})$ is the discrete configuration. What can we say if $\mathcal{A}$ has exactly two irreps: the principal and another one. We start with a homogeneous case.

Theorem (Higman)

Let $(\Omega, \mathcal{C})$ be a homogeneous c.c. If $\mathcal{A}$ has only two irreducible representations Δ_0, Δ_1, then $|\mathcal{C}| = 2$.

Proof.
$|\mathcal{C}| = 1 + n_1^2, n_1 = \dim(\Delta_1)$.
$\tau = \chi_0 + m_1\chi_1 \implies 0 = \tau(S) = k_S + m_1\chi_1(S) \implies \chi_1(S) = -\frac{k_S}{m_1} \implies \chi_1(S) \leq -1$. Hence $0 = \chi_1(J) = \chi_1(I) + \sum_{S \in \mathcal{C}, S \neq 1} \chi_1(S) \leq n_1 - (|\mathcal{C}| - 1) = n_1 - n_1^2 \implies n_1 = 1$.
□

Corollary

If $|\mathcal{C}| \leq 5$, then $\mathcal{C}$ is commutative, and, therefore, homogeneous.

Coherent configuration of a symmetric block design

Let $(\Omega_1, \Omega_2, S), S \subseteq \Omega_1 \times \Omega_2$ be a $2-(v,k,\lambda)$ symmetric block design. Let D denote the adjacency matrix of S. Then

$$JD = DJ = kJ, DD^\top = D^\top D = kI + \lambda(J-I),$$

where I and J are the identity and all-one matrices of order v.
The partititon
$\mathcal{C} = \{1_{\Omega_1}, \Omega_1^2 \setminus 1_{\Omega_1}, 1_{\Omega_2}, \Omega_2^2 \setminus 1_{\Omega_2}, S, \Omega_1 \times \Omega_2 \setminus S, S^*, \Omega_2 \times \Omega_1 \setminus S^*\}$
is a co.co. of order $2v$ and rank 8.
Its adjacency matrix has the following form

$$\begin{bmatrix} aI + b(J-I) & cD + d(J-D) \\ zD^\top + w(J-D^\top) & xI + y(J-I) \end{bmatrix}$$

Irreducible representations of the adjacency algebra

The principal representation

$$\Delta_0\left(\begin{bmatrix} aI+b(J-I) & cD+d(J-D) \\ zD^\top+w(J-D^\top) & xI+y(J-I) \end{bmatrix}\right)=$$

$$\begin{bmatrix} a+b(v-1) & ck+d(v-k) \\ zk+w(v-k) & x+y(v-1) \end{bmatrix}.$$

The corresponding irreducible subspace: $V_0=\langle \underline{\Omega_1}, \underline{\Omega_2}\rangle$.
Let $\mathbf{u}\in\mathbb{C}^v$ be a non-zero column vector of size v with zero coordinate sum. Denote $e_2=\binom{\mathbf{0}}{\mathbf{u}}$, $e_1=\binom{D\mathbf{u}}{\mathbf{0}}$.

Proposition

The subspace $\langle e_1, e_2\rangle$ is an irreducible $\mathbb{C}[\mathcal{C}]$ module.

Irreducible representations of the adjacency algebra

Proof.

$$\begin{bmatrix} aI + b(J-I) & cD + d(J-D) \\ zD^\top + w(J-D^\top) & xI + y(J-I) \end{bmatrix} \begin{pmatrix} D\mathbf{u} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} (a-b)D\mathbf{u} \\ (z-w)(k-\lambda)\mathbf{u} \end{pmatrix}$$

$$= (a-b)e_1 + (z-w)(k-\lambda)e_2$$

$$\begin{bmatrix} aI + b(J-I) & cD + d(J-D) \\ zD^\top + w(J-D^\top) & xI + y(J-I) \end{bmatrix} \begin{pmatrix} \mathbf{0} \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} (c-d)D\mathbf{u} \\ (x-y)\mathbf{u} \end{pmatrix}$$

$$= (c-d)e_1 + (x-y)e_2$$

□

Irreducible representations of the adjacency algebra

Proposition

The adjacency algebra has two irreps Δ_0 and Δ_1 where

$$\Delta_1\left(\begin{bmatrix} aI + b(J-I) & cD + d(J-D) \\ zD^\top + w(J-D^\top) & xI + y(J-I) \end{bmatrix}\right) =$$

$$\begin{bmatrix} a-b & c-d \\ (z-w)(k-\lambda) & x-y \end{bmatrix}$$

The multiplicity of Δ_1 in the standard representation is $v-1$.

Restriction to the fibers

Let $\Omega_1, ..., \Omega_f$ be the fibers of a c.c. $\mathcal{X} = (\Omega, \mathcal{C})$, $\mathcal{A}$ the adjacency algebra of $\mathcal{X}$;
$\mathcal{X}_i := (\Omega_i, \mathcal{C}^{ii})$ be the homogeneous component, $\mathcal{A}_i$ the adjacency algebra of $\mathcal{X}_i$;
The algebra $\mathcal{A}_i$ is isomorphic to $I_{\Omega_i}\mathcal{A}I_{\Omega_i}$. In what follows we identify $\mathcal{A}_i$ with $I_{\Omega_i}\mathcal{A}I_{\Omega_i}$.
Let $\Delta_0, ..., \Delta_\ell$ be irreps of $\mathcal{A}$, V_a, E_a and χ_a are the irreducible module, the primitive idempotent and the character corresponding to the irrep Δ_a

Restriction to the fibers

Theorem

Let $\Delta_a, V_a, E_a, \chi_a$ be as above. Then

1. $V_{ia} := I_{\Omega_i} V_a \neq 0 \implies V_{ia}$ is an irreducible $\mathcal{A}_i$-module; $V_{ia} \neq 0 \iff E_a I_{\Omega_i} \neq 0$;
2. $\{V_{ia} \mid V_{ia} \neq 0\}_{a=0,\ldots,\ell}$ is a complete set of irreducible $\mathcal{A}_i$ representations; $\{I_{\Omega_i} E_a \mid I_{\Omega_i} E_a \neq 0\}$ is a complete set of central primitive idempotents of $\mathcal{A}_i$;
3. If $V_{ia} \neq 0$, then the multiplicity of V_{ia} in the decomposition of $\mathbb{C}\Omega_i$ is equal to the multiplicity of V_a in the decomposition of $\mathbb{C}\Omega$.

Let us denote by Δ_{ia} the irrep of $\mathcal{A}_i$ corresponding to V_{ia} and $n_{ia} := \dim(V_{ia})$.

Restriction to the fibers

Theorem

1. $\forall_{i\in[1,f]}\ n_{i0} = 1$.
2. $\forall_{i\in[1,f]}\ \sum_{a=0}^{\ell} m_a n_{ia} = |\Omega_i|$.
3. $\forall_{a\in[0,\ell]}\ \sum_{i=1}^{f} n_{ia} = n_a$.
4. $\forall_{i,j\in[1,f]}\ \ |\mathcal{C}^{ij}| = \sum_{a=0}^{\ell} n_{ia}n_{ja}$. Equivalently, $(|\mathcal{C}^{ij}|) = N^\top N$ where $N = (n_{ia})$.
5. the matrix $(|\mathcal{C}^{ij}|)_{i,j\in[1,f]}$ is positive semidefinite.
6. $|\mathcal{C}^{ij}|^2 \leq |\mathcal{C}^{ii}||\mathcal{C}^{jj}|$ and $|\mathcal{C}^{ij}|^2 = |\mathcal{C}^{ii}||\mathcal{C}^{jj}| \implies |\Omega_i| = |\Omega_j|$.
7. If $\mathcal{C}^{ii}$ and $\mathcal{C}^{jj}$ are commutative, then $|\mathcal{C}^{ij}| \leq |\mathcal{C}^{ii}|, |\mathcal{C}^{jj}|$

Restriction to the fibers

Theorem

1. $\forall_{i \in [1,f]}\ n_{i0} = 1$.
2. $\forall_{i \in [1,f]}\ \sum_{a=0}^{\ell} m_a n_{ia} = |\Omega_i|$.
3. $\forall_{a \in [0,\ell]}\ \sum_{i=1}^{f} n_{ia} = n_a$.
4. $\forall_{i,j \in [1,f]}\ \ |\mathcal{C}^{ij}| = \sum_{a=0}^{\ell} n_{ia} n_{ja}$. Equivalently, $(|\mathcal{C}^{ij}|) = N^\top N$ where $N = (n_{ia})$.
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7. If $\mathcal{C}^{ii}$ and $\mathcal{C}^{jj}$ are commutative, then $|\mathcal{C}^{ij}| \leq |\mathcal{C}^{ii}|, |\mathcal{C}^{jj}|$

Theorem

Assume that all fibers of a co.co. have at least two elements. Then it has two irreducible representations iff it corresponds to a system of linked