Representation theory of coherent configurations/algebras

M. Muzychuk

Ben-Gurion University of the Negev,
Israel

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Conjugate representation

Proposition

Let E_i be a primitive central idempotent corresponding to Δ_i. Then

1. $E_i^\top$ and $\overline{E_i}$ are primitive c. idempotents of $\mathcal{A}$;
2. $\overline{E_i}^\top = E_i$ and $\chi_i(R^*) = \overline{\chi_i(R)}$;

If Δ_i is an irrep of $\mathcal{A}$, then $\Delta_i(A^\top)^\top$ is an irreducible representation of $\mathcal{A}$ corresponding to the idempotent $E_i^\top$. It is called the conjugate of Δ_i, notation $i^* = i$. It follows from $E_i^\top = \overline{E_i}$ that the representations $\overline{\Delta_i}$ and Δ_{i^*} are equivalent.

Block-diagonal matrix representation

Theorem (Higman)

One can choose orthonormal bases B_a of the irreducible modules V_a in such a way that

1. $[\Delta_a(A^*)]_{B_a} = [\Delta_a(A)]^*_{B_a}, A \in \mathcal{A}$.
2. $V_a \cong V_b \implies [\Delta_a(A)]_{B_a} = [\Delta_a(A)]_{B_b}, A \in \mathcal{A}$;
3. Let $B_0, ..., B_\ell$ denote orthonormal bases of pairwise inequivalent irreducible $\mathcal{A}$-subspaces $V_0, ..., V_\ell$. Denote $U = [B_0, \underbrace{B_1 \cdots B_1}_{m_1}, \cdots, \underbrace{B_\ell \cdots B_\ell}_{m_\ell}] \in M_\Omega(\mathbb{C})$.

   Then U is unitary and for each $A \in \mathcal{A}$

   $$U^*AU = \mathrm{diag}\big[[\Delta_0(A)]_{B_0}, [\Delta_1(A)]_{B_1} \otimes I_{m_1}, ..., [\Delta_\ell(A)]_{B_\ell} \otimes I_{m_\ell}\big].$$

A second basis of a coherent algebra

Let us abbreviate the (i,j)-entry of $\Delta_a(A)$ in the basis B_a as $\Delta_{(a,i,j)}(A)$, i.e. $\Delta_{(a,i,j)}(A) = (\Delta_a(A)e_i, e_j)$.
Denote by $E_{a,i,j} \in \mathcal{A}$ an element satisfying

$$U^* E_{a,i,j} U = \operatorname{diag}\left[O, O_{n_1} \otimes I_{m_1}, \ldots, \underbrace{E_{ij} \otimes I_{m_a}}_{a}, \ldots, O_{n_\ell} \otimes I_{m_\ell}\right].$$

Then $A = \sum_{(a,i,j)} \Delta_{(a,i,j)}(A) E_{a,i,j}$. The elements $E_{a,i,j}$ form a second basis of $\mathcal{A}$.

Proposition

1. $\overline{E_{a,i,j}}^{\top} = E_{a,j,i}$;
2. $\left(E_{a,i,j}, E_{a',i',j'}\right) = \delta_{a,a'}\delta_{i.i'}\delta_{j,j'} m_a$

Schur relations

Let $\Lambda := \{(a,i,j) \mid 1 \leq i,j \leq n_a\}, m_{(a,i,j)} = m_a$.

Theorem

1. $\sum_{\lambda \in \Lambda} m_\lambda \Delta_\lambda(S)\overline{\Delta_\lambda(S')} = |S|\delta_{S,S'}$.
2. $\sum_S |S|^{-1}\Delta_\lambda(S)\overline{\Delta_{\lambda'}(S)} = m_a^{-1}\delta_{\lambda,\lambda'}$.

Define the square matrix $P \in M_{\Lambda \times \mathcal{C}}$ as $P_{\lambda S} := \Delta_\lambda(S)$. Then the rows of the matrix are orthogonal w.r.t the weight $(|S|^{-1})_{S \in \mathcal{C}}$ while the columns are orthogonal w.r.t. the weight $(m_\lambda)_{\lambda \in \Lambda}$.
In a matrix form Schur's relations look as follows:

$$PD^{-1}\overline{P}^\top = M^{-1}.$$

where $D_{ST} = \delta_{S,T}|S|, M_{\lambda\mu} = \delta_{\lambda,\mu} m_\lambda$.

Character table

Orthogonality relations for irreducible characters

1. $\sum_S |S|^{-1} \chi_a(S) \overline{\chi_b(S)} = \delta_{a,b} \frac{n_a}{m_a}$;
2. If $\mathcal{C}$ is homogeneous, then $\sum_S k_S^{-1} \chi_a(S) \overline{\chi_b(S)} = \delta_{a,b} |\Omega| \frac{n_a}{m_a}$;
3. If $\mathcal{C}$ is homogeneous and χ_a is non-principal irreducible character, then $\sum_S \chi_a(S) = 0$;
4. If $\mathcal{C}$ is homogeneous and χ_a is non-principal irreducible character, then $\sum_S \frac{|\chi_a(S)|^2}{k_S} = |\Omega| \frac{n_a}{m_a}$.

Note that character table is not a square matrix, it has size $|\mathrm{Irr}(\mathcal{C})| \times |\mathcal{C}|$.

A "trivial" example

The CT of the regular scheme of S_3

	1	(12)	(13)	(23)	(123)	(132)
χ_0	1	1	1	1	1	1
χ_1	1	-1	-1	-1	1	1
χ_2	2	0	0	0	-1	-1

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Group theoretical CT of S_3

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χ_0	1	1	1
χ_1	1	-1	1
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Reduction to a square CT

Theorem

Let $\mathcal{X} = (\Omega, \mathcal{S})$ be a homogeneous co.co. Define an equivalence relation $\sim$ on $\mathcal{S}$ (the columns of the CT) as follows $\forall_{\chi \in \mathrm{Irr}(S)} S \sim T \iff \frac{\chi(S)}{k_S} = \frac{\chi(T)}{k_T}$. Then the number of $\sim$-equivalence classes is at least $|\mathrm{Irr}(\mathcal{S})|$. The equality holds iff the center $Z(\mathcal{A})$ is a coherent algebra, i.e. $Z(\mathcal{A})$ is $\circ$-closed.

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The CT of a feasible non-commutative rank six scheme

m_a		S_0	S_1	S_2	S_3	S_4	S_4^*
1	χ_0	1	10	10	20	20	20
20	χ_1	1	1	1	-7	2	2
30	χ_2	2	2	2	5	-2	-2

The center is a coherent algebra: $S_0, S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_4^*, S_3$.

Frame number

Definition

The ratio $\mathcal{F} = \dfrac{\prod_{C \in \mathcal{C}} |C|}{\prod_{i=1}^{\ell} m_i^{n_i^2}}$ is called the Frame number of the co.co.

Theorem

The Frame number of a co.co. is a rational integer.

Lemma

Let $\chi = \chi_0 + \ldots + \chi_\ell$ be the sum of all irreducible characters of $\mathcal{A}$. Then

$$\overline{P}^\top P = [\chi(S^* T)]_{S,T \in \mathcal{C}} \, .$$

Proof of the Lemma

$$(\overline{P}^{\top} P)_{S,T} = \sum_{(a,i,j)\in\Lambda} \overline{\Delta_{(a,i,j)}(S)}\Delta_{(a,i,j)}(T) =$$

$$\sum_{(a,i,j)\in\Lambda} \Delta_{(a,j,i)}(S^*)\Delta_{(a,i,j)}(T) = \sum_{(a,i,j)\in\Lambda} \Delta_a(S^*)_{j,i}\Delta_a(T)_{i,j} =$$

$$\sum_{a,j}(\Delta_a(S^*)\Delta_a(T))_{j,j} = \sum_a \chi_a(S^* T) = \chi(S^* T).$$

Galois group

Let $\mathbb{F}$ be an extension of $\mathbb{Q}$ obtained by adding all the roots of minimal polynomial of $A(S), S \in \mathcal{C}$. Then $\mathbb{F}$ is a Galois extension of $\mathbb{Q}$ and $\chi(S) \in \mathbb{F}$ for all $\chi \in \mathrm{Irr}(\mathcal{C}), S \in \mathcal{C}$. Let G be the Galois group of the extension $[\mathbb{F} : Q]$. The group G permutes the characters of the scheme.

Proposition

Galois-conjugate characters have the same dimension and multiplicity.

Proposition

The character $\chi = \sum \chi_a$ is G-invariant $\implies \chi(R) \in \mathbb{Z}$ for all $R \in \mathcal{C}$. The matrix $[\chi(S^* T)]_{S,T \in \mathcal{C}}$ has integral entries.

Frame number

Proof of Frame's Theorem

By Schur's relations

$$PD^{-1}\overline{P}^{\top} = M^{-1}.$$

Therefore

$$\det(PD^{-1}\overline{P}^{\top}) = \det(M^{-1}) \implies \det(\overline{P}^{\top}P) = \mathcal{F}.$$

By the previous Lemma $\det(\overline{P}^{\top}P) = \det(\chi(S^*T)) \in \mathbb{Z}$. $\square$

Frame number of an association scheme

Theorem

Let $(\Omega, \mathcal{S} = \{S_0, ..., S_{r-1}\})$ be an association scheme with irreducible representations $\Delta_0, ..., \Delta_\ell$. Then $n^{r-2}\frac{k_0 \cdots k_{r-1}}{m_0 m_1^{n_1^2} \cdots m_\ell^{n_\ell^2}}$ is an integer.

Theorem (Hanaki)

The adjacency algebra $\mathcal{A}$ defined over a field $\mathbb{F}$ is semisimple iff $\mathrm{char}(\mathbb{F})$ is coprime to the Frame number of a scheme.

Theorem

If $m_1 = ... = m_\ell$, then $k_1 = ... = k_{r-1} = m_1$ and the scheme is commutative

Hanaki-Uno Theorem: An association scheme of prime order p is commutative.

Proposition

Let $A \in M_n(\mathbb{F})$ be an arbitrary matrix s.t. $\mathrm{Tr}(A^k) = 0$ for each $k \in \mathbb{N}$. Then

1. if $\mathrm{char}(F) = 0$, then 0 is the only eigenvalue of A;
2. if $\mathrm{char}(F) > 0$, then every non-zero eigenvalue of A has multiplicity divisible by $\mathrm{char}(F)$.

Lemma (Hanaki)

Let $(\Omega, \mathcal{S})$ be a scheme on p points. Then

1. the characteristic polynomial of $A(S), S \in \mathcal{S}$ is congruent to $(x - k_S)^p$ modulo p;
2. $\theta(S) \equiv \theta(1)k_S \pmod{p}$ for any rational character θ of $\mathcal{A}$.

Proof of Hanaki-Uno Theorem

Let $A_0 = I, A_1, ..., A_d$ be the standard basis of $\mathcal{A}$.
Let θ be the sum of all irreducible characters Galois conjugate to $\theta_1 \in \mathrm{Irr}(\mathcal{A})$.
We have $\theta(A_i) = \theta(1)k_i - u_i p$ with $u_i \in \mathbb{Z}$. Note that $k_0 = 1, u_0 = 0$.
By orthogonality relations $\sum_{i=0}^{r-1} \theta(A_i) = 0 \implies \theta(1) = \sum_{i=1}^{r-1} u_i$.
Since the scheme is primitive, a non-principal eigenvalue λ of $A_i, i > 0$ satisfies $|\lambda| < k_i \implies |\theta(A_i)| < \theta(1)k_i$.
Therefore $u_i > 0 \implies u_i$ is a positive integer. Thus $\theta(1) = u_1 + ... + u_{r-1} \implies \theta(1) \geq r - 1$.
The character θ is a sum of Galois-conjugate characters: $\theta = \theta_1 + ... + \theta_s$. Hence $\theta(1) = s\theta_1(1)$ and we obtain $s\theta_1(1) \geq r - 1 \geq s\theta_1(1)^2$. Hence $\theta_1(1) = 1, s = r - 1$. $\square$

Known schemes of prime degree

Theorem

If $\mathcal{X} = (\Omega, \mathcal{S})$ is Schurian, then it is a Cayley scheme over $\mathbb{Z}_p$ corresponding to the Schur partition formed by the orbits of some $H \leq \mathbb{Z}_p^*$. If $|H| < p - 1$, then $\mathrm{Aut}(\mathcal{X})$ coincides with $H \ltimes \mathbb{Z}_p, H \leq \mathbb{Z}_p^*$ in its natural action on $\mathbb{Z}_p$ in a natural way: $x^{(a,b)} = ax + b$.

The only known non-Schurian schemes of prime order are those which arise from skew-symmetric Hadamard matrices of order $p + 1, p \equiv 3 (\mathrm{mod}\ 4)$. They are anti-symmetric.

Question

Are there non-schurian schemes of prime order of rank more than three?

Conjecture

Every scheme of prime order p and rank r is algebraically isomorphic to the cyclotomic scheme with the same order and rank.