Symmetries and combinatorics of finite antilattices

Tomaž Pisanski
University of Primorska and Institute of Mathematics, Physics and Mechanics , Slovenia
tomaz.pisanski(at)upr.si

This is joint work with Karin Cvetko Vah

Antilattices, known also as rectangular quasilattices, form one of the simplest varieties of non-commutative lattices. They are defined as follows: An *anitlattice* is a set N with a pair of binary operations *meet* an *join*, $\wedge$, $\vee$, which are associative, idempotent and satisfy the two absorption laws below:

$$\begin{array}{ll} x \wedge (y \wedge z) = (x \wedge y) \wedge z, & x \vee (y \vee z) = (x \vee y) \vee z, \\ x \wedge x = x, & x \vee x = x, \\ x \wedge y \wedge x = x, & y \vee x \vee y = y. \end{array}$$

In this talk we will explore the combinatorics of finite antilattices via their generating matrices. We will also investigate their substructures, congruences, symmetries, and in particular, their connection with orthogonal latin squares. The inspiration comes from a paper by Jonathan Leech [3]. *Quasiltattices* are defined by the lines above except that the last line is replaced by the formula: $x \wedge y \wedge x = x \iff y \vee x \vee y = y$. A *congruence* on N is equivalence relation that is compatible both with $\vee$ and $\wedge$. A quasilattice is *simple* if it has no non-trivial congruences. The importance of anitlattices lies on the one hand in the Laslo-Leech decomposition theorem [2], which states that a quasilattice is a lattice of antilattices and on the other hand on the fact that a simple quasilattice is either a lattice of an antilattice. Laslo-Leech decomposition theorem is a quasilattice version of the Clifford-McLean theorem for bands, [1,4] We call an antilattice that has no non-trivial sub-algebras *elementary* and antilattices without sub-algberas of even order, *odd*. Among other things, we will describe the relationships between simple, elementary and odd antilattices and the connection of the latter to pairs of orthogonal latin squares. We perform enumeration of various pseudo-varieties of finite antilattices. In particular, we focus our attention to regular antilattices as natural generalization skew antilattices. Recall that a *band* is a semigroup of idempotents. Since each antilattice is obtained as a pair of rectangular bands, we investigate rectangular bands, too and describe some of their properties. When dealing with the symmetry we point out the richness of antilattices. While the automorphism group of a rectangular band must act transitively on the set of its elements, automorphisms of antilattices may give rise to more than one orbit of elements. However, it may happen that an antilattice is transitive. In this respect non-commutative lattices may exhibit more symmetry than ordinary, commutative lattices. In addition to the case of self-dual structures, when the structure is isomorphic to the one in which the operations $\vee$ and $\wedge$ are reversed, the non-commutativity allows us to consider structures that are isomorphic to the ones in which the order of operands is reversed. If time permits we will demonstrate our computer program for working with antilattices.

Acknowledgments. The work has been supported by the ARRS Grant P1-0294 and by the bilateral ARRS Grant BI-RU/14-15-006 between the University of Primorska, Slovenia and the Sobolev Institute of Mathematics, Siberian Branch of Russian Academy of Sciences, Novosibirsk.

References

[1] A.H. Clifford, Bands of Semigroups, *Proc. Amer. Math. Soc.* **5** (1954), 499–504.

[2] G. Laslo, J. Leech, Green's equivalences on noncommutative lattices, *Acta Sci. Math. (Szeged)* **68** (2002), 501–533.

[3] J. Leech, Magic Squares, Finite Planes and Simple Quasilattices *Ars Combinatoria* **77** (2005) 75–96.

[4] D. McLean, Idempotent semigroups, *Amer. Math. Monthly* **61** (1954), 110–113.