Construction of pairs of orthogonal latin cubes based on combinatorial designs

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A *latin square* of order n is an $n \times n$ array of n symbols in which each symbol occurs exactly once in every row and in every column. A 3-dimensional array satisfying the same condition is called a *latin cube*. Two latin squares are *orthogonal* if, when they are superimposed, every ordered pair of symbols appears exactly once. Two latin cubes are orthogonal if every pair of latin squares corresponding to 2-dimensional faces of cubes are orthogonal.

Let Q be a set of cardinality n. A subset C of Q^d is called an *MDS code* (of order n with code distance $t+1$ and with length d) if $|C \cap \Gamma| = 1$ for each t-dimensional face Γ. It is well known that pairs of orthogonal latin squares are equivalent to MDS codes with code distance 3 and length 4, and pairs of orthogonal latin cubes are equivalent to MDS codes with code distance 3 and length 5.

For any prime power n, $n \geq 4$, there exist linear MDS codes of order n with code distance 3 and length 5. Consequently, there exist pairs of orthogonal latin cubes of order n. Construction based on Cartesian product provides pairs of orthogonal latin cubes of order $n_1 n_2$ if pairs of orthogonal latin cubes of orders n_1 and n_2 exist. In [4] Wilson's type construction ([1]) was used to obtain pairs of orthogonal latin cubes of order $n = 16(6k \pm 1) + 4$. Then it remains to consider orders $2k_1$ and $3k_2$ (in part) for which $\gcd(k_1, 2) = 1$ and $\gcd(k_2, 3) = 1$.

A *Steiner system* with parameters τ, d, n, $\tau \leq d$, written $S(\tau, d, n)$, is a set of d-element subsets of Q (called *blocks*) with the property that each τ-element subset of Q is contained in exactly one block. Recently Keevash [2,3] showed that the natural divisibility conditions are sufficient for existence of Steiner system apart from a finite number of exceptional n given fixed τ and d. Moreover, he proved that for any $S(\tau, d, n)$ design there exists $S(\tau + 1, d, n)$ design that contains the first design if n is sufficiently large and arithmetic conditions hold.

In [5] MDS codes are used to construct Steiner quadruple systems $S(3, 4, n)$. Now we present a construction of orthogonal latin cubes based on combinatorial designs.

Proposition. *If designs D_1 of type $S(2, 5, n)$ and D_2 of type $S(3, 5, n)$ exist and $D_1 \subset D_2$, then there exits a pair of orthogonal latin cubes of order n.*

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References

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