On uniform partitions of F^n into Hamming codes

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A code C is called *perfect binary single-error-correcting code* (briefly a perfect code) if for any vector x from the set F^n of all binary vectors of length n there exists exactly one vector $y \in C$ at the Hamming distance not more than 1 from the vector x. *The automorphism group of any partition* $P^n = \{C_0, C_1, \dots, C_n\}$ of the set F^n into perfect codes $C_0, C_1, \dots, C_n$ of length n, $\bigcup_{i=0}^{n} C_i = F^n$ is defined as the group of isometries of F^n preserving the partition P^n. A partition P^n is called *transitive*, if for any two codes C_i and C_j, i, j from $I = \{0, 1, \dots, n\}$, there is an automorphism σ from $\mathrm{Aut}(P^n)$ such that $\sigma(C_i) = C_j$. A partition P^n of F^n is defined to be 2-*transitive*, if for any two subsets $\{i_1, i_2\}$ and $\{j_1, j_2\}$ of I there exists an automorphism σ from $Aut(P^n)$ such that $\sigma(C_{i_t}) = C_{j_t}, t = 1, 2$. By definition any 2-transitive partition is transitive. A survey concerning partitions and all other necessary definitions can be found in [1].

A perfect linear code is called the *Hamming code*. Let e_i be a binary vector in F^n of weight 1 with one in the ith coordinate position. A partition $P^n = \{H_0, H_1 + e_1, \dots, H_n + e_n\}$ of F^n into cosets of Hamming codes $H_0, H_1, \dots, H_n$ of length n we call *uniform* if any two Hamming codes H_i, H_j, $i, j \in I$, satisfy $\eta_n = |H_i \bigcap H_j| = \mathrm{const}$. Such partitions of F^n into cosets of Hamming codes with the smallest possible size of η_n were constructed for length $n = 7$ by Phelps in [2] and for any $n = 2^m - 1$ for odd $m > 3$, using the Gold function by Krotov in [3].

We give the recursive construction of the class of uniform partitions into Hamming codes exploiting the classical Mollard's construction for perfect codes [4] and the results [2, 3].

Theorem. *For any* $n = 2^m - 1$, $m > 2$ *and* $l = 1, 2, \dots, [(m+1)/2]$, *with the exception* $m = 4$, $l = 1$, *there exists 2-transitive uniform partition* $P^n = \{H_0, H_1 + e_1, \dots, H_n + e_n\}$ *of* F^n *into cosets of Hamming codes* $H_0, H_1, \dots, H_n$ *of length* n *for* η_n *satisfying*

$$\log_2(\eta_n) = n - 2m + 2l - \delta(m),$$

where $\delta(m) = \begin{cases} 1 \textit{ for } m \equiv 1 (mod\ 2); \\ 0 \textit{ for } m \equiv 0 (mod\ 2). \end{cases}$

Remark. It should be noted that this theorem covered a half part of possible values of the numbers η_n. Another part is still open.

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References

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