On the numbers of transversals and multiplexes in iterated quasigroups

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A d-ary quasigroup of order n is a d-ary operation over a set of cardinality n such that the Cayley table of the operation is a d-dimensional latin hypercube of order n. Given a binary quasigroup G, the d-iterated quasigroup $G^{[d]}$ is a d-ary quasigroup that is a $(d-1)$-time composition of G with itself.

A k-multiplex K in a d-dimensional latin hypercube Q of order n or in the corresponding d-ary quasigroup is a multiset of kn entries such that each hyperplane and each symbol of Q is covered by exactly k elements of K. 1-multiplexes are mostly known as transversals. Denote by $P_k(Q)$ the number of k-multiplexes in a latin hypercube Q.

We propose a method for counting and constructing all k-multiplexes in iterated quasigroups $G^{[d]}$, assuming that a binary quasigroup G is given. The main consequence of this method is the following theorem.

Theorem 1. *Let G be a binary quasigroup of order n and let $Q(G^{[d]})$ be the d-dimensional latin hypercube corresponding to the d-iterated quasigroup.*

1. *For all odd d the d-dimensional latin hypercube $Q(G^{[d]})$ has a k-multiplex. If for some even d' the hypercube $Q(G^{[d']})$ has a k-multiplex then for all $d \geq d'$ the hypercube $Q(G^{[d]})$ has a k-multiplex.*
2. *There exists a constant $c(G,k) > 0$ for which*

$$\lim_{d\to\infty} \frac{P_k\left(Q(G^{[d]})\right)}{\left(\frac{(kn)!}{k!^n}\right)^{d-1}} = c(G,k),$$

where all d are such that $Q(G^{[d]})$ has a k-multiplex.

The asymptotic behavior of the maximum number of transversals in latin hypercubes of fixed dimension and large order was found in [2,3]. In [1] it was proved that if for an abelian group G the latin hypercube $Q(G^{[d]})$ has a transversal then for large n and fixed d the number of transversals in $Q(G^{[d]})$ asymptotically reaches the upper bound. Theorem 1 implies that the analogous statement holds for the Cayley tables of d-iterated quasigroups of fixed order and large dimension.

In this talk we also characterize a typical k-multiplex in an iterated quasigroup and provide limit constants $c(G,1)$ for the numbers of transversals in several iterated quasigroups of small orders.

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References

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