Inverse limits of m-sprouts and topological self-similar dendrites

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Definition. *Let $I = \{1, ..., m\}$ be the index set and $\Gamma = (V, E)$ be a tree such that*
1) *V is divided into 2 parts: $V = B \sqcup W$, $E \subset B \times W$; $\#B \geq m$ and the set of endpoints $B_F \subset B$;*
2) *there is injective map $\nu : I \to B$, and edge coloring $\varphi : E \to I$, injective on each $E(w)$ for any $w \in W$.*

Then the tree $\Gamma = \Gamma(B, W, E, \nu, \varphi)$, is called a m-sprout. Such settings allow to define a composition operation $\Gamma_1 * \Gamma_2$ on the set $Sp(m)$ of all m-sprouts.

There are several objects associated with a m-sprout Γ:
a) connected finite acyclic non-Hausdorff space $X(\Gamma) = (V, \tau)$, where τ is a topology generated by neighbourhoods of "black"points $\{N(b), b \in B\}$ in Γ;
b) a digraph $\mathcal{G}(\Gamma)$ with the vertex set I, called *index diagram of* Γ;
c) two semigroups G_ψ and G_ϕ of maps $\psi_w : I \cup \{0\} \to I \cup \{0\}$(resp. $\phi_w : I \to I$) relating indexed points in $\nu(I)$ to edge indices in $E(w)$.
For $u \in G_\psi$ or $u \in G_\phi$, we define $Inv(u) = \max\{I' \subset I : u(I') = I'\}$.

If $\Gamma = \Gamma_1 * \Gamma_2$, then for each $w \in W_1$ we have an isomorphic embedding $f_w : \Gamma_2 \to \Gamma$ such that $\Gamma = \bigcup\limits_{w \in W_1} f_w(\Gamma_2)$, which restricts to $f_w : X(\Gamma_2) \to X(\Gamma)$, giving the representation $X(\Gamma) = \bigcup\limits_{w \in W_1} f_w(X(\Gamma_2))$. There are also a natural embedding $J : B(\Gamma_1) \to B(\Gamma)$ and projection $\pi : X(\Gamma) \to X(\Gamma_1)$ such that $\pi \circ J = Id|_{B_1}$.

Let Γ_1 be a m-sprout, and put $\Gamma_n = \Gamma_1^n$, and denote by $X_n = X(\Gamma_n)$ the associated topological space. Consider the sequence of projections: $X_1 \xleftarrow{\pi_{1,1}} X_2 \xleftarrow{\pi_{2,1}} ... \xleftarrow{\pi_{n-1,1}} X_n \xleftarrow{\pi_{n,1}} ...$ and let its inverse limit be $X = \varprojlim X_n$. Under certain conditions on the semigroup G_ψ, X is Hausdorff. The space X satisfies the equation $X = \bigcup\limits_{w \in W_1} f_w(X)$, so X is self-similar with respect to the system $\mathcal{S} = \{f_{w_i}, w_i \in W_1\}$. Since all X_n are acyclic connected quasi-compact spaces, the same is true for X. We prove that X is a dendrite, find the conditions of finiteness of its ramification order and show that the arcs, connecting the points in $\nu(I) \subset B$, are the components of an attractor of a graph-directed system:

Theorem 1. *If for any $u \in G_\psi$, then $\#Inv(u) \leq 1$ X is a dendrite.*

Theorem 2. *If the index diagram of Γ does not contain cyclic vertices with outgoing ramification order ≥ 2, then the ramification order for the points of X is bounded.*

Theorem 3. *For any $b, b' \in \nu(I)$ there are unique s and s-tuples $j_1, ..., j_s$, $k_1, ..., k_s$ and $l_1, ..., l_s$ so that*

$$\gamma_{bb'} = \bigcup_{i=1}^{s} f_{w_{j_i}}(\gamma_{b_{k_i} b_{l_i}})$$

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References

[1] J. Charatonik, W. Charatonik, Dendrites. *Aportaciones Mat. Comun.* **22** (1998) 227–253.

[2] M. Samuel, A. V. Tetenov, D. A. Vaulin, Self-Similar Dendrites Generated by Polygonal Systems in the Plane. *Sib. Electron. Math. Rep.* **14** (2017) 737–751.