

Alphabet lifting construction of equitable partitions of Hamming graphs

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Let $G = (V, E)$ be an undirected graph. A partition (C_1, \dots, C_t) of the set V is an *equitable partition* if for all $i, j \in \{1, \dots, t\}$ any vertex of C_i has exactly m_{ij} neighbors in C_j . A matrix $M_{t \times t} = (m_{ij})$ is called the *matrix of parameters* of this partition.

The Hamming graph $H(n, q)$ is a graph whose vertices are all words of length n over the alphabet $\{0, 1, \dots, q-1\}$. Two vertices are adjacent if and only if they differ in exactly one coordinate position.

Equitable partitions of Hamming graphs were investigated by D.G. Fon-Der-Flaass in case $q = 2$ [1–3]. In this work we propose new construction of equitable partitions of Hamming graphs $H(n, q)$ for $q > 2$.

Theorem 1 (Alphabet lifting construction). *Let (C_1, \dots, C_t) be an equitable partition of the graph $H(n, q_1)$ with the matrix of parameters M . Define the partition of vertices (C'_1, \dots, C'_t) of the graph $H(n, q_1 q_2)$ as follows:*

$$\begin{aligned} \forall i \in \{1, \dots, t\} \quad \forall x = (x_1, x_2, \dots, x_n) \in H(n, q_1 q_2) \\ (x_1, x_2, \dots, x_n) \in C'_i \iff (x_1 \bmod q_1, \dots, x_n \bmod q_1) \in C_i. \end{aligned}$$

Then (C'_1, \dots, C'_t) is an equitable partition with the matrix of parameters $q_2 M + n(q_2 - 1)E$, where E is the identity matrix of order t .

Traditionally equitable 2-partitions (partitions with $t = 2$) are of special interest. Some constructions from [1] can be easily generalized for the case $q > 2$. Combining these constructions with the one from the theorem we obtain equitable 2-partitions with new matrices of parameters.

Corollary 1. *Take arbitrary $k, m', m'', b, c \in \mathbb{N}$ and prime p such that*

$$m'' < m', \quad b + c = kp^{m'}, \quad \gcd(b, c) = kp^{m''}, \quad \gcd(k, p) = 1.$$

Let k_1 be an arbitrary natural divisor of k and s be an arbitrary natural divisor of one of the numbers $m', m' - 1, \dots, m' - m''$. Then there exist $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$ there exist an equitable 2-partition of the graph $H(n, q)$ for $q = k_1 p^s$ with the matrix of parameters $\begin{pmatrix} n(q-1) - b & b \\ c & n(q-1) - c \end{pmatrix}$.

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References

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