On the volume of a compact hyperbolic antiprism

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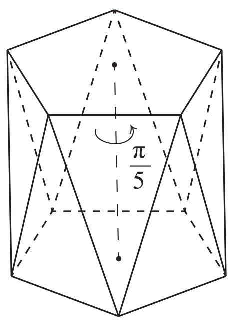
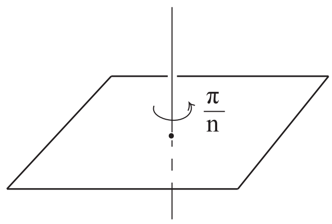
We consider a compact hyperbolic antiprism. It is a convex polyhedron with $2n$ vertices in $\mathbb{H}^3$ which has a symmetry group S_{2n} generated by a mirror-rotational symmetry of order $2n$, i.e. rotation to the angle π/n followed by a reflection. We establish necessary and sufficient conditions for the existence of such polyhedra in hyperbolic space $\mathbb{H}^3$. Then we find relations between their dihedral angles and edge lengths in the form of a cosine rule. Finally, we obtain exact integral formulas expressing the volume of a hyperbolic antiprism in terms of the edge lengths.



n=5



Theorem 1. *A compact hyperbolic antiprism with $2n$ vertices and edge lengths a, c having a symmetry group S_{2n} is exist if and only if*

$$1 + \operatorname{ch} a - 2 \operatorname{ch} c + 2\,(1 - \operatorname{ch} c) \cos \frac{\pi}{n} < 0.$$

Theorem 2. *The volume of a compact hyperbolic antiprism with $2n$ vertices and edge lengths a, c having a symmetry group S_{2n} is given by the formula*

$$V = n \int_{c_0}^{c} \frac{t\,(\operatorname{ch} a - 1)(1 + \operatorname{ch} a + 2 \operatorname{ch}^2 t - 4 \operatorname{ch} t \cos \frac{\pi}{n}) + 2\,a\,(\operatorname{ch} t - \cos \frac{\pi}{n}) \operatorname{sh} a \operatorname{sh} t}{(2 \operatorname{ch}^2 t - 1 - \operatorname{ch} a)\sqrt{R}}\, dt,$$

where $R = 1 - \operatorname{ch} a\,(2 + \operatorname{ch} a) + 2 \operatorname{ch}^2 t + 4\,(\operatorname{ch} a - 1) \operatorname{ch} t \cos \frac{\pi}{n} - 2\,(ch^2 t - 1) \cos \frac{2\pi}{n}$ *and* c_0 *is the root of the equation* $1 + \operatorname{ch} a - 2 \operatorname{ch} c + 2\,(1 - \operatorname{ch} c) \cos \dfrac{\pi}{n} = 0$.

In particular case $n = 3$ an antiprism become an octahedron with $\overline{3}$-symmetry. In this case theorems 1 and 2 are coincide with the results given in [1]. When $n = 2$ the upper and lower n-gonal faces of an antiprism degenerate to line segments. Thus we get a tetrahedron with a symmetry group S_4. The latter case was studied in [2].

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References

[1] N. V. Abrosimov, E. S. Kudina, A. D. Mednykh, On the volume of a hyperbolic octahedron with $\overline{3}$-symmetry. *Proceedings of the Steklov Institute of Mathematics* **288** (2015) 1–9.

[2] N. V. Abrosimov, Vuong Huu Bao, The volume of a hyperbolic tetrahedron with symmetry group S_4. *Trudy Instituta Matematiki i Mekhaniki UrO RAN* **23**:4 (2017) 7–17.