Combinatorial games on graphs, Coxeter-Dynkin diagrams, and the geometry of root systems

Norman J. Wildberger
University of New South Wales, Sydney, Australia
n.wildberger(at)unsw.edu.au

Coxeter-Dynkin graphs feature prominently in dozens of topics in modern mathematics, including Lie algebras and Lie groups, reflection groups, regular polytopes, lattice theory, singularities, root systems, von Neumann algebras, quantum groups, knot theory, and many areas of combinatorics.

The ADE graphs and their affine variants are particularly intriguing. Their close connections to symmetries of various kinds has even prompted some physicists to consider E_8 as a unifying object for a theory of everything.

Explaining this remarkable ubiquity of rather simple graphs is a tantalising problem, perhaps first clearly enunciated by Arnold in 1976.

In this talk we consider the graphs as central, and explore two remarkable games, the Numbers game and the Mutation game, that generate quite a lot of associated mathematics around ADE diagrams and their affine variants. The Numbers game was introduced by Moses and also studied by Eriksson. The Mutation game is roughly dual to the Numbers game. Both are played with populations (functions on vertices) on a general graph, and are very simple to define.

We will show that using only elementary analysis of these games already produces a lot of the rich theory that surround these objects. Remarkable lattices and posets will figure, Coxeter/Weyl groups make a natural appearance, the geometry of root systems and connections with representation theory of Lie groups and Lie algebras will figure prominently, and we will also present at least one intriguing challenge.