Group fusion power of association schemes

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There are many ways to construct new association schemes from smaller schemes, say direct product [1], wreath product [2], semidirect product [3,4], crested product [5], wedge product [6], etc. In most cases, the new association scheme is imprimitive. One way to construct new primitive association scheme is the symmetric power (extension) [7]. We generalize the construction of symmetric power, which involves a subgroup G of the symmetric group, and we show that the G-power scheme (defined later) is primitive in most cases.

Let $\mathfrak{X} = (X, \{R_\alpha\}_{\alpha\in\mathcal{I}})$ be an association scheme. Let $G \le \mathrm{Sym}(d)$ be a subgroup of the symmetric group of degree d. The group G acts on $\mathcal{I}^d$ naturally by permuting the indices, that is $\sigma(\alpha_1, \alpha_2, \ldots, \alpha_d) = (\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \ldots, \alpha_{\sigma(d)})$ for all $\sigma \in G$ and $(\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathcal{I}^d$. We denote by $\mathcal{O} = \mathcal{I}^d/G$ the orbits under this action. We fuse the two relations in the d-th direct power $\mathfrak{X}^d$ of the association scheme $\mathfrak{X}$, if their corresponding index sequences are in the same orbit of $\mathcal{O}$.

Theorem 1. *The above construction results in a fusion scheme of $\mathfrak{X}^d$.*

We denote the obtained association scheme by $\mathfrak{X}^d/G = (X^d, \{R_\Lambda\}_{\Lambda\in\mathcal{O}})$, and call it the G-power scheme (of $\mathfrak{X}$). Accordingly, we call $\mathfrak{X}$ the base scheme. The direct power and the symmetric power (extension) of association schemes are G-power schemes for $G = \{\mathrm{Id}\}$ and $G = \mathrm{Sym}(d)$ respectively. In fact, it is straightforward from the construction to get the following theorem.

Theorem 2. *Let $H \le G \le \mathrm{Sym}(d)$, then $\mathfrak{X}^d/G$ is a fusion scheme of $\mathfrak{X}^d/H$.*

We calculate the intersection numbers of the G-power scheme. And if the base scheme is commutative, we give the Krein parameters, eigenmatrices of the G-power scheme as well. The formulas are omitted here, which are simple combinations of those of power schemes and fusion schemes. Then we give the necessary and sufficient condition for the G-power scheme being primitive.

Theorem 3. *Let $\mathfrak{X}$ be an association scheme and let G be a subgroup of $\mathrm{Sym}(d)$. The G-power scheme $\mathfrak{X}^d/G$ is primitive if and only if G is transitive and $\mathfrak{X}$ is primitive and not thin.*

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