Phase spaces and kernel spaces of transformation semigroups

Yinfeng Zhu
Shanghai Jiao Tong University, Shanghai, China
fengzi(at)sjtu.edu.cn

This is joint work with Yaokun Wu

Let Ω be a set and $s \in \Omega^\Omega$. We write $\overline{s}$ and s^{-1} for the two maps from 2^Ω to 2^Ω such that $\overline{s}(X) = \{s(x) : x \in X\}$ and $s^{-1}(X) = \{y : s(y) = x\}$ for all $X \in 2^\Omega$. If a is an element and F is a set such that $f(a)$ is well-defined for every $f \in F$, we adopt the notation $F(a)$ for the set $\{f(a) : f \in F\}$.

A transformation semigroup is a pair (S, Ω) where Ω is a set and S is a sub-semigroup of the full transformation semigroup Ω^Ω. The **phase space** of (S, Ω) is the transformation semigroup $(\overline{S}, 2^\Omega)$ where $\overline{S} = \{\overline{s} : s \in S\}$. The **reduced phase space** of (S, Ω) is the transformation semigroup $(\overline{S}, \overline{S}(\Omega))$.

Let $\mathrm{P}(\Omega)$ be the set of all partitions of Ω. For all $s \in \Omega^\Omega$, we write $\tilde{s}$ for the map from $\mathrm{P}(\Omega)$ to $\mathrm{P}(\Omega)$ such that $\tilde{s}(\Pi) = \{s^{-1}(\pi) : \pi \in \Pi\} \setminus \{\emptyset\}$ for each $\Pi \in \mathrm{P}(\Omega)$. The **kernel space** of (S, Ω) is the transformation semigroup $(\tilde{S}, \mathrm{P}(\Omega))$ where $\tilde{S} = \{\tilde{s} : s \in S\}$. Let 0_Ω be the partition of Ω into singleton sets. The **reduced kernel space** of (S, Ω) is the transformation semigroup $(\tilde{S}, \tilde{S}(0_\Omega))$.

We study some problems about phase spaces and kernel spaces. Below are some sample results in this ongoing research.

Theorem 1. *Let (S, Ω) be a transformation semigroup and let p and q be two positive integers such that $p + q \leq |\Omega|$ and $p \geq q$. If $\overline{S}$ acts transitively on $\binom{\Omega}{p}$, then $\overline{S}$ acts transitively on $\binom{\Omega}{q}$.*

Theorem 2. *There is a polynomial-time algorithm to decide whether or not a given transformation semigroup is a reduced phase space.*

We have a counterpart of **Theorem 2** for reduced kernel space under some assumptions. We also propose parameters to measure the deviation of a transformation semigroup from being a reduced phase space or a reduced kernel space.