Permutation representations of finite groups and association schemes (Part 1)

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Contents

Permutation Groups

1. A permutation group is a subgroup of the symmetric group $\mathrm{Sym}(\Omega)$, where Ω is a finite set.
2. More generally, an action $(G; \Omega)$ of a finite group G on a set Ω is a homomorphism
$$G \to \mathrm{Sym}(\Omega).$$

The essential difference between the above two is faithfulness:

If the kernel of the action is trivial, then G can be identified with its image which is a subgroup of Ω.

Notation for action $(G; \Omega)$

We denote the image of $\omega \in \Omega$ under $g \in G$ by

$$\omega^g.$$

So we multiply permutations from left to right.

If G is a finite group and H is a subgroup of G, then the coset action $(G; G/H)$ is

$$G \to \mathrm{Sym}(G/H)$$

defined by

$$g \mapsto (Hx \mapsto Hxg)$$

Note: $G/H = \{Hx \mid x \in G\}$.
If $H = 1$, then this is the right regular representation.

What is a representation?

Right regular representation is

$$G \to \mathrm{Sym}(G).$$

A permutation representation of G is nothing but an action.
A (matrix) representation of G is a homomorphism

$$G \to GL_n(\mathbb{F})$$

for some field $\mathbb{F}$. A representation of G is a homomorphism

$$G \to GL(V)$$

for some vector space V.

The word representation means a description using a typical object (in this case, $\mathrm{Sym}(\Omega)$ and $GL_n(\mathbb{F})$).

Basic concepts

Let $(G;\Omega)$ be an action. The G-orbit of $\omega \in \Omega$ is

$$\omega^G = \{\omega^g \mid g \in G\}.$$

The stabilizer of ω is

$$G_\omega = \{g \in G \mid \omega^g = \omega\}.$$

Then

$$|\omega^G| = |G : G_\omega|.$$

For the coset action $(G; G/H)$, the stabilizer of $Hx \in G/H$ is $x^{-1}Hx$, so the kernel of the coset action (homomorphism) is the core of H:

$$\mathrm{Core}_G(H) = \bigcap_{x \in G} x^{-1}Hx \lhd G.$$

Transitivity

An action $(G; \Omega)$ is

1. transitive if $\Omega = \omega^G$,
2. semi-regular if $G_\omega = 1$ $(\forall \omega \in \Omega)$,
3. regular if transitive and semi-regular.

The right regular representation is regular.

Action by conjugation $(G; G)$ is

$$\begin{array}{rcl} G & \to & \mathrm{Sym}(G) \\ g & \mapsto & (h \mapsto g^{-1}hg). \end{array}$$

The orbits are the conjugacy classes of G, so this action is not transitive unless $|G| = 1$.

Equivalence

Two actions $(G; \Omega)$ and $(G; \Delta)$ are equivalent if there exists a bijection $f : \Omega \to \Delta$ such that the diagram

$$\begin{array}{ccc} \Omega & \xrightarrow{f} & \Delta \\ {\scriptstyle g}\big\downarrow & & {\scriptstyle g}\big\downarrow \\ \Omega & \xrightarrow{f} & \Delta \end{array}$$

commutes for all $g \in G$, i.e., $f(\alpha^g) = f(\alpha)^g$ $(\forall g \in G, \forall \alpha \in \Omega)$.

If $(G; \Omega)$ is a transitive action, then it is equivalent to the coset action $(G, G/G_\omega)$.

Every action is equivalent to a "disjoint union" of coset actions.

Permutation matrices

Denote by $M_\Omega(\mathbb{F})$ the $\mathbb{F}$-algebra of matrices with entries in a field $\mathbb{F}$, whose rows and columns are indexed by a finite set Ω.

For $g \in \mathrm{Sym}(\Omega)$, the permutation matrix representing g is

$$P_g = (\delta_{\alpha^g,\beta})_{\alpha,\beta\in\Omega} \in M_\Omega(\mathbb{F}).$$

This gives a (matrix) representation of $\mathrm{Sym}(\Omega)$:

$$\begin{array}{rcl} \mathrm{Sym}(\Omega) & \to & GL_{|\Omega|}(\mathbb{F}) \\ g & \mapsto & P_g \end{array}$$

If $(G;\Omega)$ is an action, then composing it with the above representation of $\mathrm{Sym}(\Omega)$, one obtains a (matrix) representation $G \to GL_{|\Omega|}(\mathbb{F})$, called a permutation representation (!?).

Representations

A representation of G is a homomorphism $G \to GL(V)$, where V is a vector space over a field $\mathbb{F}$. If $(G; \Omega)$ is an action, then

$$G \to \mathrm{Sym}(\Omega)$$
$$G \to \mathrm{Sym}(\Omega) \to GL_{|\Omega|}(\mathbb{F}) \cong GL(\mathbb{F}^{\Omega})$$

are both called a permutation representation.

$$\begin{aligned} \text{action } (G;\Omega) &\iff \text{permutation representation } G \to \mathrm{Sym}(\Omega) \\ &\implies \text{permutation representation } G \to GL_{|\Omega|}(\mathbb{F}) \\ &\implies \text{matrix representation } G \to GL_{|\Omega|}(\mathbb{F}) \\ &\implies \text{representation } G \to GL(\mathbb{F}^{\Omega}) \end{aligned}$$

Equivalent representations

Two representations $\phi : G \to GL(V)$ and $\psi : G \to GL(W)$ are equivalent if there is an $\mathbb{F}$-linear isomorphism $f : V \to W$ such that the diagram

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\phi(g)"'] & W \arrow[d, "\psi(g)"'] \\ V \arrow[r, "f"] & W \end{tikzcd}$$

commutes for all $g \in G$, i.e., $f(\phi(g)v) = \psi(g)f(v)$
$(\forall g \in G,\ \forall v \in V)$.

Equivalences

Equivalent permutation representation are equivalent as representations.

$$\begin{tikzcd} \Omega \arrow[r, "f"] \arrow[d, "g"'] & \Delta \arrow[d, "g"'] \\ \Omega \arrow[r, "f"] & \Delta \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} \mathbb{F}^\Omega \arrow[d, "P_g^\Omega"'] & \textcolor{red}{\mathbb{F}^\Delta} \arrow[l, "f^*"'] \arrow[d, "P_g^\Delta"'] \\ \mathbb{F}^\Omega & \mathbb{F}^\Delta \arrow[l, "f^*"'] \end{tikzcd}$$

Indeed

$$f^* P_g^\Delta(u) = (u_{f(\alpha)^g})_{\alpha\in\Omega} = (u_{f(\alpha^g)})_{\alpha\in\Omega} = P_g^\Omega f^*(u)$$

for all $u \in \mathbb{F}^\Delta$.

Classification

Given a finite group G,

(i) Classify permutation representations of G.

(ii) Classify (matrix) representations of G.

(i) is equivalent to the knowledge of all subgroups. (ii) is more difficult. (i) cannot be reduced to (ii).
Note

$$\{\text{representations}\} \supset \{\text{permutation representations}\}$$

but equivalence is different.

The Mathieu group M_{23} has two inequivalent permutation representation on sets Ω, Δ with $|\Omega| = |\Delta| = 253$, such that the corresponding matrix representations on $\mathbb{C}^\Omega$ and $\mathbb{C}^\Delta$ are equivalent.

The Goal

Our goal of studying permutation groups is to reveal the structure of the algebra

$$\mathbb{C}[\{P_g \mid g \in G\}] \subseteq M_\Omega(\mathbb{C}).$$

The full knowledge of this algebra for the right regular representation is equivalent to that of all representations of G.

In general,

$$\mathbb{C}[\{P_g \mid g \in G\}] \cong \bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C}),$$

and for the right regular representation, this means that G has r "irreducible" representation up to equivalence, and an arbitrary representation is a direct sum of irreducibles.

Imprimitivity

A transitive action $(G;\Omega)$ is imprimitive if there is a nontrivial G-invariant partition of Ω. Otherwise it is said to be primitive. We say a partition trivial if it has only one part, or every part is a singleton. A nontrivial G-invariant partition is called a system of imprimitivity, and G acts also on this system.

1. If $H \lneq K \lneq G$, then the partition
$$\{\{Hx \mid x \in Ky\} \mid Ky \in G/K\}$$
is a system of imprimitivity on $(G; G/H)$.
2. An action $(G; G/H)$ is primitive iff H is a maximal subgroup of G.

Product actions and 2-orbits

If $(G;\Omega)$ and $(G;\Delta)$ are actions, then one can define

$$\begin{array}{rcl} G & \to & \mathrm{Sym}(\Omega \times \Delta) \\ g & \mapsto & ((\alpha,\beta) \mapsto (\alpha^g, \beta^g)) \end{array}$$

The orbits of the action $(G;\Omega^2)$ are called 2-orbits, or orbitals of G.
The set of 2-orbits will be denoted by Ω^2/G.
For $R \subseteq \Omega^2$, the adjacency matrix $A(R) \in M_\Omega(\mathbb{F})$ of R is

$$(A(R))_{\alpha,\beta} = \begin{cases} 1 & \text{if } (\alpha,\beta) \in R, \\ 0 & \text{otherwise.} \end{cases}$$

If $R \subseteq G^2$ is a 2-orbit of the right regular action, and if $(g_1, g_2) \in R$, then

$$(A(R))_{\alpha,\beta} = \begin{cases} 1 & \text{if } \alpha\beta^{-1} = g_1 g_2^{-1} \\ 0 & \text{otherwise.} \end{cases} = \delta_{(g_1 g_2^{-1})^{-1}\alpha,\beta}.$$

Next lecture

Proposition

Let $(G;\Omega)$ be an action. If $R \in \Omega^2/G$, then

$$A(R)P_g = P_g A(R) \quad (g \in G).$$

Proof.

$$(A(R)P_g)_{\alpha\beta} = A(R)_{\alpha,\beta^{g^{-1}}},$$
$$(P_g A(R))_{\alpha\beta} = A(R)_{\alpha^g,\beta}.$$

□

In the next lecture, I will axiomatize the properties of 2-orbits of a permutation group to introduce the notion of association scheme.