Graphs Coverings 1

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August 7, 2018

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From the very beginning we have in mind a clear analogy between the graph coverings and branched graph coverings, and their counterparts that are **useful concepts in complex analysis**, and generally, in continuos mathematics!

Why to investigate graph coverings?

General question: How to describe large (finite) objects and how to control properties of such objects.

Three essentially different approaches:

- Use induction (recursion),
- Use probabilistic method (non-constructive),
- Use of symmetries

Prominent example of the third approach is the construction of a Cayley graph! Theory of (regular) graph coverings belongs to the third approach.

Applications of graph coverings

- Solution of the Heawood map coloring problem (Ringel,Youngs,...)
- Graph embeddings (Tucker, White, Stahl,...)
- Highly symmetrical graphs (Biggs, Feng,...)
- Nowhere-zero flows (N., Skoviera,...)
- Spectral theory (Cvetkovic, Brankovic, Sato,...)
- Cages and near cages, (Exoo, Jajcay,...)
- Extremal graphs of given valency and diameter, (Siran, Miller,....)
- Classification of vertex-transitive maps on given surface, (Conder, Siran, N., Pisanski,Karabas,...)
- Enumeration of graph coverings, (Kwak, Hofmeister,...)
- Map enumeration, (Tutte, Liskovets, Walsh, Mednykh, Giorgetti, N.,...)

Main idea

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- (projection) Find a universal cover for P, usually an infinite graph $\tilde{X}$. Construct quotient(s) X by means of $p : \tilde{X} \to X$ such that P is preserved. Here factorisation by particular semiregular groups of automorphisms of $\tilde{X}$ apply.

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END OF STORY???

Two examples

- Property P_1, X is connected and 2-valent,
- Property P_2, X is connected and X is locally C_6, graph induced on the neighbours of every vertex is 6-cycle.

The beginning: Graphs based on darts

A basic problem of the classical definition: **Quotient of a graph is not a graph!**

Darts basic elements from which we build graphs are "darts" (arcs, blades,...), these objects can be interpretted as edges endwed with an (implicit) orientation,

Graphs Graph is a triple $X = (D, V; L, I)$, where $L \in Sym(D)$ is an involutory permutation (the dart-reversing involution), and $I : D \to V$ is an equivalence relation (the incidence relation). We assume $D \cap V = \emptyset$.

Category of graphs

A *homomorphism* between graphs $X_1 = (D_1, V_1; L_1, I_1)$ and $X_2 = (D_2, V_2; L_2, I_2)$ is a mapping $\psi : D_1 \cup V_1 \to D_2 \cup V_2$ such that $\psi L_1 = L_2 \psi$ and $\psi I_1(x) = I_2 \psi(x)$.

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vertices elements of V,

initial vertex of $x \in D$ just Ix,

edges orbits of L, three kinds of edges:

semiedge $\{x, Lx\} = \{x\}$, fixed points of L,

loop $I(x) = IL(x)$, but $x \neq Lx$,

ordinary edge $x \neq Lx$ and $Ix \neq ILx$,

Graph homomorphisms from groups of automorphisms

$X = (D, V; I, L)$, $G \leq Aut(X)$,
The *quotient graph* $X/G = (D', V'; I', L')$ is given by setting

- $D' = \{[x]_G | x \in D\}$, the set of orbits of G on D,
- $V' = \{[v]_G | v \in V\}$
- $I'[x]_G = [Ix]_G$,
- $L'[x]_G = [Lx]_G$.

Exercise 1. Prove that X/G is well-defined and that the projection $x \mapsto [x]$, $x \in D$ is a graph homomorphism.
Exercise 2.: Find $K_4/\langle(234)\rangle$ and the respective homomorphism as a function defined on the pairs (i, j), $i \neq j$, $i, j \in \{1, 2, 3, 4\}$.

Remarks

Simple graphs:

- L is fixed-points-free (no semiedges),
- for every x, y, $ILx \neq Ix$ (no loops),
- $Ix = Iy$ implies $ILx \neq ILy$, (no parallel edges).

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Homomorphisms are determined by the images of darts!

Graphs based on darts (arcs) alternatively

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D a set of darts, $\approx$ an equivalence (incidence) relation, L is an (dart-reversing) involution

Category Graphs Graphs with morphisms, $p : X_1 \to X_2$,
$pL_1 = L_2 p$, edges are preserved
$p([x]_\approx) \subseteq [p(x)]_\approx$, incidence between edges and vertices is preserved,

Vertices $V = \{[x]_\approx | x \in D\}$, no vertices of degree 0, if $v = [x]_\approx$, then $[x]_\approx$ is the set of darts incident with v,

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This definition is equivalent with the one used by Serre: Trees (1980), or in the theoretical physics (Kontsevich and others).

Graph coverings

Graph covering: a graph morphism $p : X \to Y$, s.t. for every $x \in D$ the restriction $p|_{[x]}$ is a **bijection**, sometimes we use the notation $[x] = D_v$, for the set of all darts based at a fixed vertex v,

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An alternative point of view: the fibre $p^{-1}(x)$ intersects each set $[y] \in p^{-1}[x]$ exactly in one dart, **unique dart lifting property**

Important Note: If Y is connected then p is surjective,

In most considerations we are restricted to **connected** graphs

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- there is an **asymmetry** in graph theory: while for invariants based on vertices the definition of a graph based at vertices is appropriate,
- for investigation of graph invariants based on edges (say the flow theory, or edge-colourings) this definition is not a proper one!
- Book: Graph homomorphisms by Hell and Nesetril

Cayley graph, very important example

darts $D = G \times X$, G a group, $X = X^{-1}$ a (multi)set of generators,

vertices $V = G$,

incidence $I(g, x) = g$,

dart-reversing inv. $L(g, x) = (g, x)^{-1} = (gx, x^{-1})$.

Theorem

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projection $p\colon (g, x) \mapsto x$ is a covering $Cay(G, X) \to B(X)$,

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Two actions of a group on the vertices of a Cayley graph

two actions of G:

Left-multiplication by elements of G gives an embedding $G \to Aut(Cay(G, X))$ and the embedded copy of G acts regularly on each fibre over a dart and over the vertex w!

Right multiplication by an element $h \in G$ permutes the vertices ion the unique fiber over w, it will be called the *monodromy action*!

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Question: When the left and right actions coincide?

Exercise 3. Consider the 3-sided prism as a Cayley graph based on Z_6 and D_3. Determine the two actions for the dihedral case.

Cayley voltage graph

Let $Y = (D, V; L, \mathcal{I})$ be a (connected) graph and G be a group: we introduce a mapping $\nu : D \to G$ (called an Cayley voltage assignment) satisfying $\nu(Lx) = (\nu(x)^{-1})$.
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Moreover, the left multiplication by elements of G defines a subgroup $G_{left} \leq Aut(X)$ of automorphisms of the derived graph permuting the vertices in the fibres of G !

Examples

Exercise 4. Determine the Generalised Petersen graphs as derived graphs based on cyclic groups.

Exercise 5. Determine the n-dimensional cubes as a derived graph based on the elementary abelian 2-group of rank n.

Exercise 6. Prove that the derived graph defined by the next figure has diameter two.

Example: The Hoffman-Singleton graph

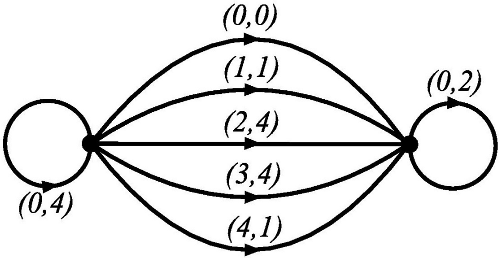

Figure: Siran, Siagiova,The Hoffman-Singleton graph derived from the voltage assignment in $\mathbb{Z}_5 \times \mathbb{Z}_5$