Permutation representations of finite groups and association schemes (Part 2)

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G2R2, Novosibirsk

Review

In the previous lecture, we discussed group actions, permutation representations (in two different meanings), and 2-orbits.
For an action $(G;\Omega)$ and $R \in \Omega^2/G$, we have seen

$$A(R)P_g = P_g A(R) \quad (g \in G).$$

The converse is also true in the following sense. Let

$$V_{\mathbb{F}}(G,\Omega) = \{X \in M_\Omega(\mathbb{F}) \mid XP_g = P_g X \ (\forall g \in G)\}$$

be the centralizer algebra of G. Then

$$V_{\mathbb{F}}(G,\Omega) \text{ has a basis } \{A(R) \mid R \in \Omega^2/G\}.$$

Indeed,

$$X \in V_{\mathbb{F}}(G,\Omega) \iff (X)_{\alpha,\beta} = (X)_{\alpha^g,\beta^g} \quad (\forall \alpha,\beta \in \Omega,\ g \in G).$$

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The algebra $V_{\mathbb{F}}(G,\Omega)$

Since $\{A(R) \mid R \in \Omega^2/G\} = \{A(R_1), \ldots, A(R_r)\}$ is a basis of the algebra $V_{\mathbb{F}}(G,\Omega)$, there exist $p_{ij}^k \in \mathbb{F}$ such that

$$A(R_i)A(R_j) = \sum_{k=1}^{r} p_{ij}^k A(R_k). \tag{1}$$

We also have

$$\forall i,\ \exists i',\ A(R_i)^\top = A(R_{i'}), \tag{2}$$

$$\sum_{k=1}^{r} A(R_k) = J \text{ (all-one matrix)}, \tag{3}$$

if, moreover, the action is transitive,

$$\exists i_1,\ A(R_{i_1}) = I. \tag{4}$$

(We may assume $A(R_1) = I$.)

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Association schemes

A family $\{R_1, \ldots, R_r\}$ of subsets of Ω^2 is an association scheme if

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\begin{aligned}
A(R_i)A(R_j) &= \sum_{k=1}^{r} p_{ij}^k A(R_k) \quad (\text{for some } p_{ij}^k \in \mathbb{Z}), \\
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Notation

With the notation

$$\begin{aligned} 1_\Omega &= \{(\alpha,\alpha) \mid \alpha \in \Omega\}, \\ R^\top &= \{(\beta,\alpha) \mid (\alpha,\beta) \in R\} \quad (R \subseteq \Omega^2), \end{aligned}$$

we see

$$A(R_1) = I \iff R_1 = 1_\Omega,$$
$$A(R_i)^\top = A(R_i^\top),$$

while

$$\sum_{k=1}^{r} A(R_k) = J \iff \bigcup_{k=1}^{r} R_k = \Omega^2 \quad \text{(disjoint)}.$$

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$$R_1 = 1_\Omega.$$

What is p_{ij}^k? If $(\alpha, \beta) \in R_k$, then

$$\begin{aligned} p_{ij}^k &= \sum_{\gamma\in\Omega} (A(R_i))_{\alpha,\gamma}(A(R_j))_{\gamma,\beta} \\ &= |\{\gamma \in \Omega \mid (\alpha,\gamma) \in R_i,\ (\gamma,\beta) \in R_j\}|. \end{aligned}$$

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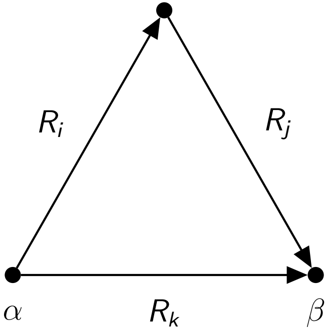
Intersection number

$$p_{ij}^{k} = \sum_{\gamma \in \Omega} (A(R_i))_{\alpha,\gamma} (A(R_j))_{\gamma,\beta}$$
$$= |\{\gamma \in \Omega \mid (\alpha,\gamma) \in R_i,\ (\gamma,\beta) \in R_j\}|$$
$$= \#\gamma$$



The adjacency algebra

If $\mathcal{R} = \{R_1, \ldots, R_r\}$ is an association scheme, then

$$\mathbb{F}[\mathcal{R}] = \mathrm{Span}\{A(R_i) \mid i = 1, \ldots, r\}$$

is the adjacency algebra of $\mathcal{R}$.

It is closed under multiplication, transpose. Also under the entrywise (Schur–Hadamard) product

$$(A \circ B)_{\alpha,\beta} = (A)_{\alpha,\beta}(B)_{\alpha,\beta}.$$

Right regular representation

If $R \subseteq G^2$ is a 2-orbit of the right regular action $(G_R; G)$, and if $(g_1, g_2) \in R$, then

$$(A(R))_{\alpha,\beta} = \begin{cases} 1 & \text{if } \alpha\beta^{-1} = g_1 g_2^{-1} \\ 0 & \text{otherwise.} \end{cases} = \delta_{(g_1 g_2^{-1})^{-1}\alpha,\beta}.$$

Thus, for $\mathcal{R} = G^2/G_R$,

$$\mathbb{F}[\mathcal{R}] = V_{\mathbb{F}}(G_R; G) = \mathrm{Span}\{A(R_g) \mid g \in G\},$$

where

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is the left regular representation.
Since $R_g R_h = R_{gh}$, $\mathbb{F}[\mathcal{R}] \cong \mathbb{F}[G]$ (group algebra).

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The Johnson scheme $J(n, 2)$

Let $G = S_n = \mathrm{Sym}(\{1, \ldots, n\})$, and let

$$\Omega = \{\alpha \mid \alpha \subseteq \{1, \ldots, n\},\ |\alpha| = 2\}.$$

Then G acts naturally on Ω. The 2-orbits are

$$\begin{aligned} R_0 &= 1_\Omega, \\ R_1 &= \{(\alpha, \beta) \mid 1 = |\alpha \cap \beta|\}, \\ R_2 &= \{(\alpha, \beta) \mid 0 = |\alpha \cap \beta|\}. \end{aligned}$$

The numbering of 2-orbits by $0, 1, 2$ reflects the distance in the Johnson graph (Ω, R_1). Writing $A(R_i) = A_i$, we have

$$\begin{aligned} A_1^2 &= 2(n-2)I + (n-2)A_1 + 4A_2, \\ (A_1 - 2(n-2)I)&(A_1 - (n-4)I)(A_1 + 2I) = 0. \end{aligned}$$

Thus $\mathbb{C}[\mathcal{R}] = \mathbb{C}[A_1] \cong \mathbb{C}^3$.

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Two bases of the algebra $\mathbb{C}[\mathcal{R}]$

- We have determined the structure of the adjacency algebra $\mathbb{C}[\mathcal{R}]$ for the Johnson scheme $J(n, 2)$.
- The structure of $\mathbb{C}[\mathcal{R}]$ can be as simple as $\mathbb{C}^3$, or as complex as $\mathbb{C}[G]$.
- But our goal is not only to determine the abstract structure (such as $\mathbb{C}^3$),
- but to find numerical specification of the distinguished basis $\{A(R) \mid R \in \mathcal{R}\}$.
- Having found an isomorphism $\mathbb{C}[\mathcal{R}] \cong \mathbb{C}^3$ means that there exists another distinguished basis E_0, E_1, E_2 such that

$$E_i E_j = \delta_{ij} E_i.$$

The subsequent lectures will focus on various aspects of the change-of-basis matrix of these two bases.

The goal of Problem Solving Session (I)

1. The 2-orbits of the group $M_{23} \leq S_{23}$ on

$$\Omega = \{\alpha \mid \alpha \subseteq \{1, \ldots, n\},\ |\alpha| = 2\}.$$

is also the Johnson scheme $J(23, 2)$,

$$\begin{aligned} V_{\mathbb{C}}(M_{23}, \Omega) &= \mathrm{Span}\{A(R_0), A(R_1), A(R_2)\} \\ &= \mathrm{Span}\{E_0, E_1, E_2\} \cong \mathbb{C}^3. \end{aligned}$$

2. The 2-orbits of M_{23} on the set of blocks $\mathcal{B}$ of the Witt system W_{23} (to be explained in the Problem Solving Session tonight) is also an association scheme,

$$\begin{aligned} V_{\mathbb{C}}(M_{23}, \mathcal{B}) &= \mathrm{Span}\{A(R'_0), A(R'_1), A(R'_2)\} \\ &= \mathrm{Span}\{E'_0, E'_1, E'_2\} \cong \mathbb{C}^3. \end{aligned}$$

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The goal of Problem Solving Session (II)

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2. The above two permutation representations of M_{23} are equivalent (as representations).

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