# Graphs Coverings 2

Roman Nedela

University of West Bohemia, Pilsen

Novosibirsk State University, Novosibirsk

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# Fundamental grupoid of a graph

A walk $W$ in a graph is a sequence of darts $W = x_0 x_1 \dots x_n$ such that $I(x_{i+1} = IL(x_i)$, (the terminal vertex of $x_i$ coincides with the initial vertex of $x_{i+1}$. $W$ is closed, if

$$I(x_0) = I(W) = IL(x_n) = IL(W).$$

Observe that the operators $I$ and $L$ naturally extend to the set of all walks.
In particular,

$$L(W) = W^{-1} = x_n^{-1} x_{n-1}^{-1} \dots x_0^{-1}$$

is the inverse walk.
Walks with the same terminal and initial vertex can be concatenated, thus we have a partially defined product of walks.

# Fundamental grupoid

The empty walk based at a vertex $v$ behaves as a neutral element. In graphs there is a canonical way to choose a representative of a homotopy class of walks. In particular, a walk is *irreducible*, if it does not contain a subsequence of the form $xx^{-1}$. Each walk determines a unique irreducible walk.

**Theorem**

*The set of irreducible walks with the product operation forms the fundamental grupoid of a graph $X$ denoted $\pi(X)$.*

## Fundamental group of a graph

**Theorem**

*The set of closed irreducible walks based at a vertex $v$ forms a group, the fundamental group of $X$ denoted $\pi(X, v)$. Moreover, if $X$ is connected then $\pi(X, v)$ and $\pi(X, u)$ are conjugate in $\pi(X)$, in particular, they are isomorphic.*

**Problem.** What is the structure of $\pi(X, v)$?

# Structure of the fundamental group

Standard approach: Take the generators defined by cotree darts with respect to a chosen spanning tree and show that the group $\pi(X, v)$ is free, however, ...

**Observation:** a closed walk consisting from a single dart $x$ such that $x^{-1} = x$ is not contractible

**Theorem**

*Let $X$ be a connected graph with $s$ semiedges, $e$ ordinary edges and loops, $v$-vertices. The fundamental group of a connected graph $X$ is a free product*

$$\pi(X) = Z * Z * \cdots * Z * Z_2 * Z_2 * \cdots * Z_2.$$

*There are $s$ $Z_2$-factors and $e - v + 1$ $Z$-factors.*

# Exercises

- Describe connected graphs with trivial fundamental group.
- Describe graphs with a finite fuundamental group.
- Describe graphs with an abelian fundamental group.

# Fundamental group of a graph and its cover

unique walk lifting lemma for every vertex $\tilde{u} \in fib(u)$ and a walk $W$ based at the vertex $u$ there is a unique lift $\tilde{W}$ based at $\tilde{u}$,

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**the monodromy action** each element of $\pi(u)$ acts as a permutation on $fib(u) = p^{-1}(u)$,

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in particular, $CT(p)$ is semiregular ($X$ is connected).

## n-fold coverings and the star-functor

**Corollary 1:** If $p : X \to Y$ is a covering between finite graphs, then there **exists** $n$ such that $|p^{-1}(y)| = |p^{-1}(v)| = n$ for every dart $y$ and every vertex $v$ of $Y$.

**Corollary 2:** if $X$ is connected, then $|CT(p)| \leq n \leq |Mon(p)|$ and $|CT(p) = n|$ if and only if the action is regular on each fibre.

**Theorem**

*Given graph covering between connected graphs $p : X \to Y$ induces an embedding $p^* : \pi(X, \tilde{v}) \to \pi(Y, p(v))$.*
*Any subgroup $H$ of finite index $n$ of the fundamental group $\pi(Y, w)$ determines a covering $X \to Y$, $X$ is connected and $\pi(X, v) \cong H$.*

# The universal cover

We always have $1 \leq \pi(X, v)$, so there is an associated covering $\tilde{X} \to X$.

If $X$ has at least one cycle, then $\tilde{X}$ is an infinite tree.

If $Y \to X$ is a covering, then $\tilde{X}$ covers $Y$.

If $X$ is a $k$-valent graph, then $\tilde{X}$ is the infinite $k$-valent infinite tree. If the degree sequence of $X$ is more complex, the tree $\tilde{X}$ is much harder to describe.

**Problem.** Let $Y \to X$ and $Z \to X$ are two coverings over $X$. Let both $Y$ and $Z$ be finite graphs. Is there a finite graph $\tilde{X}$ covering both $Y$ and $Z$?

## Characterisations of *regular* coverings:

Let $p : X \to Y$ be a covering between (connected) graphs

- $Mon(p)$ is regular on a fibre,
- $CT(p)$ is regular on a fibre,
- $Mon(p) \cong CT(p)$
- there exists $G \leq Aut(X)$ **fixed point free on vertices and darts s.t.** $Y \cong X/G$.
- there exists a **Cayley voltage assignment** $\xi : D(Y) \to G$ such that $p$ is equivalent to the natural projection $Y^{\xi} \to Y$,
- the embedding $p^*(\pi(X)) \lhd \pi(Y)$ is a normal subgroup.

## Few words on irregular covers

- instead ordinary voltages we use permutation voltages,
- necessary condition for regularity: a cycle lifts to a uniform 2-factor,
- sometimes good model to describe the monodromy action is the Schreier action diagram,

**Exercise 6.** Find examples of irregular coverings.

**Exercise 7.** Find example of an irregular covering over a one-vertex graph.