Permutation representations of finite groups and association schemes (Part 3)

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August 15, 2018
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In the previous lecture, we discussed the adjacency algebra $\mathbb{C}[\mathcal{R}]$ of an association scheme $\mathcal{R}$. In this lecture we always take $\mathbb{F} = \mathbb{C}$.

If $\mathcal{R}$ is the set of 2-orbits of an action $(G; \Omega)$, then

$$\mathbb{C}[\mathcal{R}] = V_\mathbb{C}(G, \Omega)$$

which is the centralizer in $M_\Omega(\mathbb{C})$ of

$$\{P_g \mid g \in G \}.$$ 

That is, the two algebras

$$\mathbb{C}[\mathcal{R}] \text{ and } \mathbb{C}[\{P_g \mid g \in G \}]$$

centralize each other.

Note that $\mathbb{C}[\{P_g \mid g \in G \}]$ is not a group algebra since $P_g \ (g \in G)$ are not necessarily linearly independent.

Then what is $\dim \mathbb{C}[\{P_g \mid g \in G \}]$?
For $A \in M_n(\mathbb{C})$, write $A^* = A^\top$.

**Definition**

A subalgebra $\mathcal{A}$ of $M_n(\mathbb{C})$ is a $\ast$-algebra if

$$A \in \mathcal{A} \implies A^* \in \mathcal{A}.$$ 

If $(G; \Omega)$ is an action, then $\{P_g \mid g \in G\}$ is closed under transposition: $P_g^\top = P_{g^{-1}}$. Indeed,

$$(P_g^\top)^{\alpha\beta} = (P_g)^{\beta\alpha} = \delta_{\beta g \alpha} = \delta_{\alpha g^{-1} \beta} = (P_{g^{-1}})^{\alpha\beta}.$$ 

Thus $\mathbb{C}[\{P_g \mid g \in G\}]$ is a $\ast$-algebra.

If $\mathcal{R}$ is an association scheme, then $\{A(R) \mid R \in \mathcal{R}\}$ is closed under transposition. Thus $\mathbb{C}[\mathcal{R}]$ is a $\ast$-algebra.
The standard module

Let \( \mathcal{A} \subseteq M_n(\mathbb{C}) \) be a \(*\)-algebra. Since \( \mathbb{C}^n \) is a left \( M_n(\mathbb{C}) \)-module, it is also an \( \mathcal{A} \)-module, which is called the standard module.

Recall that an \( \mathcal{A} \)-module \( W \) is irreducible if it has no nontrivial \( \mathcal{A} \)-submodule.

**Proposition**

Let \( \mathbb{C}^n \supseteq V \) be an \( \mathcal{A} \)-submodule of a \(*\)-algebra \( \mathcal{A} \subseteq M_n(\mathbb{C}) \). Then \( V^\perp \) is also an \( \mathcal{A} \)-submodule.

Thus, the standard module \( \mathbb{C}^n \) is an orthogonal direct sum of irreducible \( \mathcal{A} \)-modules. This means that, \( \exists \) unitary matrix \( U \),

\[
U^* \mathcal{A} U = \text{algebra of block diagonal matrices}
\]

and each block (say, size \( m \times m \)) acts on \( \mathbb{C}^m \) irreducibly, hence isomorphic to \( M_m(\mathbb{C}) \) by Wedderburn’s theorem.
Wedderburn’s theorem (cf. Rotman’s book)

Upon block-diagonalizing \( \mathcal{A} \), we find a block \( \mathcal{B} \subseteq M_m(\mathbb{C}) \) such that

\( \mathbb{C}^m \) is an irreducible \( \mathcal{B} \)-module, and it is faithful

\[ \Rightarrow \quad \mathcal{B} \text{ is simple (}\not\exists \text{ nontrivial ideal)} \]

\[ \Rightarrow \quad \mathcal{B} = \bigoplus_{i=1}^{k} L_i \quad \text{pairwise isomorphic minimal left ideals} \]

\[ \mathcal{B} \cong \text{End}_{\mathcal{B}}(\mathcal{B})^{op} \]

\[ \cong \text{End}_{\mathcal{B}}\left(\bigoplus_{i=1}^{k} L_i\right)^{op} \]

\[ \cong M_k(\mathbb{C})^{op} \quad (\text{by Schur’s lemma}) \]

\[ \cong M_k(\mathbb{C}) \]

\[ \Rightarrow \quad \text{the only irreducible } \mathcal{B} \text{-module has } \text{dim} = k \]

\[ \Rightarrow \quad m = k, \text{ so } \mathcal{B} = M_m(\mathbb{C}). \]
Block diagonalization

\[ U^*AU = \begin{bmatrix} \ldots & \ldots & M_m(\mathbb{C}) & \ldots & \ldots \\ \ldots & & & & \ldots \\ \ldots & & & & \ldots \\ M_m(\mathbb{C}) & \oplus & \mathbb{C}^m & \oplus & \ldots \\ \ldots & & & & M_m(\mathbb{C}) \end{bmatrix} = \mathbb{C}^n. \]

Be careful! Isomorphic modules may appear more than once. What if

\[ U^*AU = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \mid A \in M_m(\mathbb{C}) \right\}? \neq \begin{bmatrix} M_m(\mathbb{C}) & 0 \\ 0 & M_m(\mathbb{C}) \end{bmatrix}. \]

It is block-diagonal with two blocks, but \( \mathcal{A} \cong M_m(\mathbb{C}) \).
A (matrix) representation of a $\mathbb{C}$-algebra $\mathcal{A}$ is a $\mathbb{C}$-algebra homomorphism $\Delta : \mathcal{A} \to M_m(\mathbb{C})$. This is equivalent to giving an $\mathcal{A}$-module of dimension $m$.

Since we only consider $\ast$-algebras, we require a representation to be $\ast$-representation, that is

$$\Delta(A^\ast) = \Delta(A)\ast.$$

Block diagonalization by a unitary matrix satisfies this requirement, since

$$(U^\ast A U)^\ast = U^\ast A^\ast U.$$ 

Our representations are of the form “first conjugating by a unitary matrix, then extracting a block”.
Block diagonalization

The block diagonalization of a \( \ast \)-algebra \( \mathcal{A} \subseteq M_n(\mathbb{C}) \) is described as

\[
U^\ast \mathcal{A} U = \begin{bmatrix}
\cdots \\
\Delta_i(A) \\
\cdots \\
\Delta_i(A) \\
\cdots
\end{bmatrix}
\]

(\( \Delta_i(A) \) is repeated \( m_i \) times).

\( \Delta_i : \mathcal{A} \to M_{n_i}(\mathbb{C}) \) is a \( \ast \)-representation, and \( \Delta_i \) is surjective by Wedderburn’s theorem.

\[
\left\{ \begin{array}{c}
\Delta_i(A) \\
\cdots \\
\Delta_i(A)
\end{array} \right| \begin{array}{c}
A \in \mathcal{A}
\end{array} \right\} \cong M_{n_i}(\mathbb{C}).
\]
Centralizer ( = matrices which commute with given ones)

What is the centralizer of

$$U^* A U = \begin{bmatrix}
\Delta_i(A) \\
\vdots \\
\Delta_i(A)
\end{bmatrix}$$

($\Delta_i(A)$ is repeated $m_i$ times).
Centralizer of $M_n(\mathbb{C})$ is $\mathbb{C}I$ (scalar matrices).

The centralizer of

$$U^*AU = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \mid A \in M_d(\mathbb{C}) \right\}?$$

is

$$\begin{bmatrix} \mathbb{C}I & \mathbb{C}I \\ \mathbb{C}I & \mathbb{C}I \end{bmatrix} = \left\{ \begin{bmatrix} a_{11}I & a_{12}I \\ a_{21}I & a_{22}I \end{bmatrix} \mid \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_2(\mathbb{C}) \right\}. $$
Centralizer (II)

The centralizer of

\[ U^*AU = \left\{ \begin{bmatrix} A & \cdots & \cdots & A \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ A & \cdots & \cdots & A \end{bmatrix} \mid A \in M_d(\mathbb{C}) \right\} \quad (m \text{ times}) \]

in \( M_{dm}(\mathbb{C}) \) is

\[ \left\{ \begin{bmatrix} a_{11}I_d & \cdots & a_{1m}I_d \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ a_{m1}I_d & \cdots & a_{mm}I_d \end{bmatrix} \mid [a_{ij}] \in M_m(\mathbb{C}) \right\}. \]

\( M_d(\mathbb{C}) \otimes I_m \) and \( I_d \otimes M_m(\mathbb{C}) \) centralize each other.
$M_d(\mathbb{C}) \otimes I_m$ and $I_d \otimes M_m(\mathbb{C})$ centralize each other.

The centralizer of

$$
\begin{bmatrix}
\cdots \\
\Delta_i(A) \\
\cdots \\
\Delta_i(A) \\
\cdots \\
\end{bmatrix}
$$

where $\Delta_i(A)$ is repeated $m_i$ times, $A \in M_{n_i}(\mathbb{C})$, is

$$
\begin{bmatrix}
\cdots \\
I_{n_i} \otimes M_{m_i}(\mathbb{C}) \\
\cdots \\
\end{bmatrix}
$$

Centralizer(Centralizer($A$)) = $A$  (The double centralizer theorem).
The Johnson scheme $J(n, 2)$

It is the action $(S_n; \Omega)$, where

$$\Omega = \{ \alpha \mid \alpha \subseteq \{1, \ldots, n\}, |\alpha| = 2 \}.$$ 

Its adjacency algebra $\mathbb{C}[\mathcal{R}] \subseteq M_\Omega(\mathbb{C})$ is isomorphic to $\mathbb{C}^3$. The (block) diagonal form is

$$\begin{bmatrix}
\vdots \\
M_{n_i}(\mathbb{C}) \otimes I_{m_i} \\
\vdots
\end{bmatrix} = \begin{bmatrix}
\mathbb{C}I_{m_0} \\
\mathbb{C}I_{m_1} \\
\mathbb{C}I_{m_2}
\end{bmatrix}$$

$$\mathbb{C}[\{P_g \mid g \in S_n\}] \cong \begin{bmatrix}
M_{m_0}(\mathbb{C}) \\
M_{m_1}(\mathbb{C}) \\
M_{m_2}(\mathbb{C})
\end{bmatrix}$$

multiplicity-free $\iff n_i = 1 \ (\forall i).$
The group algebra $\mathbb{C}[G]$ (I)

The group algebra of a group $G$ is the algebra with basis indexed by the elements of $G$, such that multiplication is done just as in $G$. The algebra of the right regular representation $(G_R; G)$

$$\mathcal{A} = \mathbb{C}[\{P_g \mid g \in G\}]$$

is such an algebra. Since this is a $\ast$-algebra, its block diagonal form is

$$\begin{bmatrix}
\cdots \\
\Delta_i(A) \\
\cdots \\
\cdots \\
\end{bmatrix} = \begin{bmatrix}
\cdots \\
M_{n_i}(\mathbb{C}) \otimes I_{m_i} \\
\cdots \\
\cdots \\
\end{bmatrix}$$

We aim to show $m_i = n_i$. 
The group algebra $\mathbb{C}[G]$ satisfies $m_i = n_i$

\[
U^* A U = \begin{bmatrix}
\cdots \\
M_{n_i}(\mathbb{C}) \otimes I_{m_i} \\
\cdots
\end{bmatrix}
\]

The size $|G|$ of the matrix is $\sum n_i m_i$.
The dimension of the algebra is $|G| = \sum n_i^2$.
The centralizer of $A$ is the algebra of left regular representation, its dimension is

\[
|G| = \sum m_i^2.
\]

Equality in Cauchy–Schwarz inequality forces $m_i = n_i \ (\forall i)$. This proves

\[
\mathbb{C}[G] \cong \begin{bmatrix}
\cdots \\
M_{n_i}(\mathbb{C}) \otimes I_{n_i} \\
\cdots
\end{bmatrix}.
\]
The centralizer algebra $V_C(G, \Omega)$, or more generally, the adjacency algebra $\mathbb{C}[\mathcal{R}]$ of an association scheme, is (abstractly) isomorphic to

$$\bigoplus_i M_{n_i}(\mathbb{C}).$$

In terms of block diagonal matrices, $M_{n_i}(\mathbb{C})$ appear $m_i$ times in the diagonal.

In particular, the group algebra:

$$\mathbb{C}[G] \cong \text{the algebra of the right regular representation}$$
$$\cong \text{the algebra of the left regular representation}$$
$$= \text{centralizer of the right regular representation}$$

is also isomorphic to the direct sum of the full matrix algebras.