Graphs Coverings 3

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Summary 1: General (classical) staff

- For some deep reasons we have introduced a bit non-standard **category of graphs with semi-edges**,
- We have analysed the **fundamental grupoid and fundamental group** of such graphs,
- The concept of **graph covering** $p : X \to Y$,
- The **monodromy action**, **the group of covering transformations**,
- The **universal cover** over X,
- **Regular coverins**

Summary 2: Semiregular groups and voltages

- Graph **homomorphisms defined by subgroups** of $Aut(X)$,
- **Semiregular subgroups** of automorphisms v.s. **regular coverings**,
- Construction of covers over X by means of **voltages**,
- Cayley graphs and regular coverings,
- Some properties of graph coverings, Irregular vs regular covers

Example: Cyclic covers over a dipole and Fermat curves.

Example: The Petersen graph as an irregular cover over a one-vertex graph, permutation graphs.

Three popular voltage assignments

What do they have in common?

- Cayley v.a. $\tilde{V} = V \times G$, $\tilde{D} = D \times G$ and G acts on the second coordinate by right multiplication;
- permutation v.a. $\tilde{V} = V \times \{1, 2, \ldots, n\}$ and $G \leq Sym(n)$ acts on the second coordinate;
- Schreier v.a. $\tilde{V} = V \times \{Hg | g \in G$ and G acts on the second coordinate by right multiplication,

Can we unify the combinatorial approach the construction of the cover over a graph X?

Voltage as. is a morphism from $\pi(X) \to G$ and from $\pi(X, u) \to G_u$

If we have a walk $W = x_0 x_1 \dots x_n$ and a voltage assignment ξ defined on D, then we can extend it to walks by setting $\xi(W) = \xi(x_0)\xi(x_1)\dots\xi(x_n)$.

Moreover, ξ becomes a homorphism $\pi(X) \to G$!

Monodromy action and action of the local voltage group!

Generalisation to Voltage spaces

action If α be a homomorphism $G \to Sym(\Omega)$, then $x \cdot g = x \cdot \alpha(g)$ is a right action defined by the homomorphism α

voltage space A triple (F, G, α), where (F, G) is a G-space (G acts on an abstract fibre F), and $\alpha : \pi(X) \to (F, G)$ is a homomorphism from the fundamental grupoid to (F, G).

Important Note: If for each edge we choose one of the two underlying darts, thus forming a subset $D^+ \subseteq D$, then any assignment $x \mapsto g$, $x \in D$ and $g \in G$ extends uniquely to a homomorphism $\pi(X) \to (F, G)$.

The derived graph from the voltage space

The construction of the cover $\tilde{X}$: $\tilde{D} = D \times F$, $\tilde{V} = V \times F$, $\tilde{I}(x, a) = (Ix, a)$, $\tilde{L}(x, a) = (Lx, a \cdot \alpha(x))$.

The projection: $(x, a) \mapsto x$ erasing the second coordinate is a graph covering $\tilde{X} \to X$.

Homotopy (Schreier) voltage assignment

Recall: There is a correspondence between subgroups of the fundamental group and coverings.

- Take any $H \leq \pi(X, u)$ (of finite index m).
- Take any spanning tree T,
- Set $\xi(x) = 1$ if x is a tree dart, otherwise set $\xi(x) = g_x \in \pi(X, u)$, where g_x is the generator associated with the cotree dart x.
- Construct the derived graph.

Theorem

The covering $\tilde{X} \to X$ is regular.

Connectedness of the cover

The local voltage group: Is the subgroup of the voltage group generated by the voltages of closed walks.

The restriction of ξ to the local group is a group homomorphism $\xi : \pi(X, u) \to G$.

Theorem

The derived graph is connected if and only if X is connected and $G_u = G$.

Example. Homological covers.

Equivalence of coverings and T-reduced v.a.

Two coverings over X, $p_1 : \tilde{X}_1 \to \tilde{X}$ and $p_2 : \tilde{X}_2 \to X$ are **isomorphic** if there exists a graph isomorphism Φ such that $p_1 = \Phi \cdot p_2$.

An **algorithm** to compare two coverings defined voltage assignments:

- transform both coverings to equivalent T-reduced v.a. ξ_1 and ξ_2,
- reduce the voltage groups to the local groups (if needed),
- the two coverings are equivalent if there is a permutation σ such that $\xi_2 = \xi_1^\sigma$ (permutation voltages),
- the two regular coverings are equivalent if there is a **group automorphism** σ such that $\xi_2 = \sigma \cdot \xi_1$ (Cayley voltages),

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Take $H < F(\beta)$ in a free group of rank β,
$F = \langle x_1, x_2, \ldots, x_\beta \rangle$,

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Form the Schreier voltage space (F, G, α) by setting $F = \{Hy | y \in G\}$, $G = \pi(X, u)$ with the action defined by the right multiplication. The assignment on the darts is defined by setting $\alpha(x) = x$,

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But H is the fundamental group of the covering graph, thus H is free.

What is the rank of $H < F_\beta$ if $[F_\beta : H] = m$?

$\tilde{X} \to X$ is an m-folded covering, over a one-vertex graph with β loops, thus $|\tilde{V}| = m$, $|\tilde{E}| = \beta \cdot m$,

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$H = \pi(\tilde{X}, \tilde{v})$, thus rank of H is $|\tilde{E}| - |\tilde{V}| + 1 = m\beta - m + 1 = m(\beta - 1) + 1$

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