# Graphs Coverings 4

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- Proof that the girth is large:
- Observe that $X$ covers $T$ and is by definition vertex-transitive,
- at each vertex of $X$ we have a copy of $T$,
- It follows that girth of $X$ is at least $2d+1$

# Proof by Maltsev theorem 1

Input two positive integers $(k, m)$ such that $\frac{1}{k} + \frac{1}{m} \leq \frac{1}{2}$.

- Take the k-valent tiling of the hyperbolic plane by regular $m$-gons, observe the underlying graph is an infinite graph satisfying the required property
- Let $\Delta(k, m, 2)$ be the group of orientation preserving automorphisms, it is a 2-generator group defined by the presentation:

$$\langle a, b | a^k = b^2 = (ab)^m = 1 \rangle$$

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- $\Delta(k, m, 2)$ is a matrix group,

# Proof by Maltsev theorem 2

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# Proof by Maltsev theorem

- Maltsev: finitely generated matrix groups are residually finite, in particular, $\Delta(k, m, 2)$ is res. fin.
- geometric meaning: For every radius $r$ there exists a subgroup $\Gamma_r \lhd \Delta$ of **finite index**
- such that for each dart $x$ inside $D_r$, every $g \neq 1$ takes $x$ outside $D_r$, in particular $\Gamma_r$ is semiregular on darts and vertices,
- the projection $X \to X/\Gamma_r$ is a regular covering mapping $D_r$ bijectively,
- $\Gamma_r \lhd \Delta$ the factor group $\Delta(k, m, 2)/\Gamma_r$ acts regularly on darts of the quotient,
- Consequence: Arround every vertex there is a copy of $D_r$, and the number of darts is the index $[\Delta : \Gamma_r]$.

# Application 3. Heawood problem case $n = 12s + 7$

We follow the ideas by published in the book by Ringel and Youngs, with the modification taken from Goddyn, Richter, Siran, published in JCT B 2007,

- the Heawood bound
- preliminary analysis, coverings and duality,
- the Kirkhoff law(s), voltage-current duality,
- v.a. in $Z_{12s+7}$ on the Ringel graph $H$ from gracefull labelling of a path on $2s + 1$ vertices,
- one-face embedding of $H$,

# Application 3. Heawood problem case $n = 12s + 7$

Goddyn, Richter, Siran, JCT B 2007:

Construction of the complete nowhere-zero flow on the Ringel graph using a graceful labelling:

- Let $P = P_{2s+1}$ be a path with $2s + 1$ vertices, and $f : V \to \{1, 2, \ldots, 2s + 1\}$ be a graceful labelling,
- Embedd $P$ into the equator of the sphere, form a triangulation $T$ by adding two vertices $a$, $b$ at the north- and south-pole, join them by an edge, and for each vertex $w$ of $P$ add the edges $aw$ and $bw$,
- form a labelling $f'$ of vertices of $T$ by setting $f'(a) = 6s + 4$, $f'(b) = 1$, and $f'(w) = 3f(w)$ for every $w$ in $P$,
- Let $x = uv$ be a dart of $T$, we set $\xi(x) = f'(v) - f'(u)$,

# Application 3. Heawood problem case $n = 12s + 7$

Properties of $\xi$:

Construction of the complete nowhere-zero flow on the Ringel graph using a graceful labelling:

- Observation 1: $\xi$ takes all the non-zero values from the integer interval $[-6s-3, 6s+3]$, and each appears exactly once,
- Observation 2: $\xi$ **is an iteger-valued function on darts derived from** $f'$ **determining vertex-potentials, it takes value zero on each cycle**,
- Taking $\xi$ modulo $n = 12s + 7$ we get a v.a. in $Z_n$ satisfying the above properties,
- **Use of duality:** The dual $H = T^*$ is the Ringel graph, and $\xi$ transfers to $\xi^*$ on $H$ satisfying the Kirkhoff law for each vertex.

## Degree-diameter problem

Problem: Input $k$-the valency, diameter 2. Wanted: $k$-valent graphs of diameter two with the number of vertices as close to $k^2+1$ as possible.

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- Let $q = 4s+1$ be a prime power, take $GF(q)$,
- Let $X$ be a two vertex graph, $V = \{u, v\}$, such that $u$ and $v$ are joined by $4s+1$ parallel edges, and there are $s$ loops attached to each vertex.

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- Let $X$ be a two vertex graph, $V = \{u, v\}$, such that $u$ and $v$ are joined by $4s+1$ parallel edges, and there are $s$ loops attached to each vertex.
- v. a. in $(GF, +) \times (GF, +)$: For a dart $x_g$, $g \in GF(q)$ joining $u$ to $v$, set $\xi(x_g) = (g, g^2)$, for a loop dart $l_i$ set $\xi(l_i) = (0, h^{2i})$, and for a loop $l_i'$ set $\xi(l_i') = (0, h^{2i-1})$, where $h$ is a primitive element.

# Properties of the cover

- The degree is $k = 2s + q = (3q-1)/2$,
- The number of vertices is $2|G| = 2q^2 = \frac{8}{9}(k+\frac{1}{2})^2 \sim \frac{8}{9}k^2$,
- the diameter is 2!

# On the proof

By definition $(GF(q), +) \times (GF(q), +)$ acts regularly on each of the two vertex fibres. Hence it is enough to check that every vertex is reachable from $u_{(0,0)}$ and from $v_{(0,0)}$ by a path of length at most two.

In particular, $v_{(x,y)}$, where $(x, y) \neq (g, g^2)$, is reachable by a path of length two if at least one of the following two equations: $(x, y) = (g, g^2) + (0, h^{\pm(2i-1)})$, $(x, y) = (g, g^2) + (0, h^{\pm 2i})$, has a solution $(g, i)$, $g \in GF(q)$ and $i \in \{1, 2, \dots s\}$.

# Degree-diameter problem, remarks

- the graphs constructed above are called McKay-Miller-Širáň graphs,
- the above construction was discovered by Šiagiová, see JCT B 2001,
- a construction of $k$-valent graphs of diameter two with $k^2 - O(k^{3/2})$ using the construction of a Cayley graph based on affine group can be found in JCT B 102, 2012

# Flows and coverings, flow lifts to a flow!

B. Alspach, Y.-P. Liu, and C.-Q. Zhang: Cayley graphs on solvable groups admits a nowhere-zero 4-flow, SIAM J. Discrete Math.(1996)

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- 3. Using the fact that minimal normal subgroup is elementary Abelian, we observe that $X/N$ is either a $p$-cycle with a semiedge attached to each vertex, or a $p$-prism, colour edges of $X/N$ and lift the 3-edge colouring.

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- More general results N+Škoviera, Combinatorica 2001, P. Potočnik

## Some classical statements in terms of voltages

**Petersen theorem:** A bridgeless cubic graph covers a one-vertex graph with one loop and one semiedge,

Can you recognise these two statements?

**Theorem**

*Every 4-valent graph covers the bouquet of two loops.*

**Theorem**

*A planar bridgeless cubic graph covers the 3-semistar.*